ECE 171A: Linear Control System Theory
Discussion 1: Review on ODEs (I)

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Outline

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First-order linear homogeneous ODEs

In Lecture 2, we will discuss about the first-order and second-order linear ODEs. Here, we review the methods for solving first-order linear ODEs.

A first-order **homogeneous** linear ODE is of the form

$$\dot{x}(t) = ax(t)$$

where $a \in \mathbb{R}$ is a constant, $\dot{x}(t)$ denotes the derivative of $x(t)$.

- The solution is
$$x(t) = e^{at}x(0).$$
(Recall the function with first-order derivative being itself is e^x.)
- We can easily verify it by observing
$$\dot{x}(t) = ae^{at}x(0) = ax(t),$$
$$x(0) = e^{a\times 0}x(0) = x(0).$$

We give some simple examples in the next few slides.

Example 1: Stable system

Consider a first-order ODE

$$\dot{x}(t) + 2x(t) = 0, \quad x(0) = 1.$$

We have $\dot{x}(t) = -2x(t)$ implying that

$$\frac{dx(t)}{dt} = -2x(t) \Rightarrow \frac{d \ln x(t)}{dt} = \frac{1}{x(t)} \frac{dx(t)}{dt} = -2$$

by the chain rule. Now we integrate both sides,

$$\int_0^t \frac{d}{du} \ln x(u) du = \int_0^t (-2) du = -2t,$$

$$\ln \frac{x(t)}{x(0)} = -2t \Rightarrow x(t) = x(0)e^{-2t} = e^{-2t}.$$

You can verify the answer by

$$\dot{x}(t) + 2x(t) = -2e^{-2t} + 2e^{-2t} = 0$$

and $x(0) = 1$.

Example 2: stable system

Consider a first-order ODE

$$\dot{x}(t) + 2x(t) = 0, x(0) = 2$$

- It is similar to the previous problem except the different initial values are different.
- So the solution is

$$x(t) = x(0)e^{-2t} = 2e^{-2t}.$$

- You can verify the answer by

$$\dot{x(t)} + 2x(t) = -4e^{-2t} + 4e^{-2t} = 0$$

and $x(0) = 2$.

Example 3: Unstable system

Consider a first-order ODE

$$\dot{x}(t) - 2x(t) = 0, \quad x(0) = 1$$

The solution is

$$x(t) = x(0)e^{2t} = e^{2t}.$$

You can verify the answer by

$$\dot{x}(t) - 2x(t) = 2e^{2t} - 2e^{2t} = 0$$

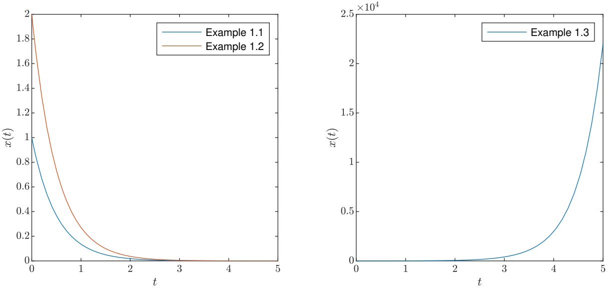
and $x(0) = 1$.

Stability

Stability can be judged by either their solutions or figures.

- The solutions of these stable systems are Ce^{-2t} which all converge to 0;
- The solution of the unstable system is $\tilde{C}e^{2t}$ which is unbounded.

Their solutions are shown in the following figure



Outline

First Order Linear Homogeneous ODEs

First Order Linear Nonhomogeneous ODEs

Nonhomogeneous ODEs

We consider the general form of a first-order linear ODE:

$$\dot{x}(t) + cx(t) = u(t), \tag{1}$$

where $c \in \mathbb{R}$ is a given constant, and $u(t)$ is a given function.

- We multiply (1) by the integrating factor $\mu(t) = e^{ct}$.
- Since $d\mu(t) = c\mu(t)dt$, (1) becomes

$$\begin{aligned}\frac{d}{dt}(\mu(t)x(t)) &= \dot{\mu}(t)x(t) + \mu(t)\dot{x}(t) \\ &= \mu(t)(\dot{x}(t) + cx(t)) \\ &= \mu(t)u(t).\end{aligned}$$

- Now we denote $\mu(t)x(t) = g(t)$ and $\mu(t)u(t) = h(t)$. Then, we have

$$\dot{g}(t) = h(t).$$

Nonhomogeneous ODEs

Thus, we integrate both sides and we get

$$\int \dot{g}(t)dt = \int h(t)dt \Rightarrow g(t) = \int h(t)dt + c_1$$

Then,

$$\mu(t)x(t) = \int e^{ct}u(t)dt + c_1$$

Thus, the general solution is:

$$x(t) = e^{-ct}\left(\int e^{ct}u(t)dt + c_1\right), \tag{2}$$

where c is the same constant in (1), and c_1 is another constant to be determined from initial conditions.

Example 4: Stable system with positive input

Consider a first-order ODE

$$\dot{x}(t) + 2x(t) = 5, x(0) = 1$$

- First, we can get the integrating factor $\mu(t)$ is $\mu(t) = e^{2t}$.
- the general solution is

$$\begin{aligned} x(t) &= e^{-2t}\left(\int 5e^{2t}dt + c_1\right) \\ &= e^{-2t}\left(\frac{5}{2}e^{2t} + c_1\right) = \frac{5}{2} + c_1 e^{-2t}, \end{aligned}$$

  where $c_1 = -\frac{3}{2}$ satisfies $x(0) = 1$.
- Note that

$$\dot{x}(t) + 2x(t) = -\frac{3}{2}(-2)e^{-2t} + 5 - 3e^{-2t} = 5.$$

Example 5: Stable system with negative input

Consider a first-order ODE

$$\dot{x}(t) + 2x(t) = -5, x(0) = 1$$

Since $\mu(t) = e^{2t}$ and

$$\int -5e^{2t}dt = -\frac{5}{2}e^{2t},$$

the solution is $x(t) = -\frac{5}{2} + \frac{7}{2}e^{-2t}$.
Also, you can check the answer by

$$\dot{x}(t) + 2x(t) = -\frac{7}{2}(-2)e^{-2t} - 5 - 7e^{-2t} = -5.$$

Review on Integration by parts

The integration by parts formula states:

- in the form of indefinite integral

$$\int u(t)\dot{v}(t)dt = u(t)v(t) - \int \dot{u}(t)v(t)dt$$

- in the form of definite integral

$$\begin{aligned}\int_a^b u(t)\dot{v}(t)dt &= [u(t)v(t)]_a^b - \int_a^b \dot{u}(t)v(t)dt \\ &= u(b)v(b) - u(a)v(a) - \int_a^b \dot{u}(t)v(t)dt\end{aligned}$$

In the Example 6 and 7, we will use the integration by parts to find antiderivative of te^{2t} and $e^{2t}\sin t$. In general, we let $\dot{v}(t) = e^{ct}$, i.e., $v(t) = \frac{1}{c}e^{ct}$.

Example 6: System with polynomial input

Consider a first-order ODE

$$\dot{x}(t) + 2x(t) = t, x(0) = 1$$

Since

$$\int te^{2t}dt = \frac{1}{2}te^{2t} - \frac{1}{2}\int e^{2t}dt = \frac{te^{2t}}{2} - \frac{e^{2t}}{4}$$

using integration by parts, the solution is

$$x(t) = e^{-2t}\left(\frac{te^{2t}}{2} - \frac{e^{2t}}{4} + c_1\right) = \frac{1}{2}t - \frac{1}{4} + c_1e^{-2t},$$

where $c_1 = \frac{5}{4}$ satisfies $x(0) = 1$. So you can check the answer by

$$\dot{x}(t) + 2x(t) = \frac{1}{2} + \frac{5}{4}(-2)e^{-2t} + t - \frac{1}{2} + \frac{5}{2}e^{-2t} = t.$$

Example 7: System with trigonometric input

Consider a first-order ODE

$$\dot{x}(t) + 2x(t) = \sin t, x(0) = 1$$

First, we compute the integral below by integration by parts.

$$\begin{aligned}
I = \int e^{2t} \sin t dt = \frac{1}{2} \int \sin t d(e^{2t}) &= \frac{1}{2} \left(e^{2t} \sin t - \int e^{2t} \cos t dt \right) \\
&= \frac{1}{2} e^{2t} \sin t - \frac{1}{4} \int \cos t d(e^{2t}) \\
&= \frac{1}{2} e^{2t} \sin t - \frac{1}{4} \left(e^{2t} \cos t - \int e^{2t} (-\sin t) dt \right) \\
&= \frac{1}{4} e^{2t} (2 \sin t - \cos t) - \frac{1}{4} \int e^{2t} \sin t dt \\
&= \frac{1}{4} e^{2t} (2 \sin t - \cos t) - \frac{1}{4} I
\end{aligned}$$

It implies

$$I = \frac{1}{1 + 1/4} \times \frac{1}{4} e^{2t} (2 \sin t - \cos t) = \frac{1}{5} e^{2t} (2 \sin t - \cos t)$$

Example 7: System with trigonometric input

Consider a first-order ODE

$$\dot{x}(t) + 2x(t) = \sin t, x(0) = 1$$

Thus, the solution is

$$x(t) = e^{-2t}\left(\frac{1}{5}e^{2t}(2\sin t - \cos t) + c_1\right) = \frac{1}{5}(2\sin t - \cos t) + c_1 e^{-2t},$$

where $c_1 = \frac{6}{5}$ satisfies $x(0) = 1$. You can check the answer by

$$\begin{aligned}\dot{x}(t) + 2x(t) &= \frac{1}{5}(2\cos t + \sin t) + \frac{6}{5}(-2)e^{-2t} \\ &\quad + \frac{1}{5}(4\sin t - 2\cos t) + \frac{12}{5}e^{-2t} \\ &= \sin t.\end{aligned}$$

Simulations of their solutions