ECE 171A: Linear Control System Theory

Discussion 5: Review on Complex numbers, rational functions, and laplace transform

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Outline

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Complex numbers

A complex number $z \in \mathbb{C}$ has a real and an imaginary part, and can be represented in either Cartesian or Polar coordinates.

- **Cartesian form**: z is represented as a linear combination of basis vectors in the complex plane, i.e., the sum of the real part and imaginary part.
- **Polar form**: z is represented by a magnitude r and phase θ.

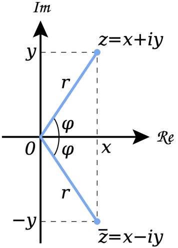

Figure: An illustration of the complex plane. The real part of a complex number $z = x + iy$ is x, and its imaginary part is y.

Operations

	Cartesian	Polar
z	$z = a + bi, \quad a, b \in \mathbb{R}$	$z = re^{i\theta}, r, \theta \in \mathbb{R}$
$\bar{z}$ Conjugate	$\bar{z} = a - bi$	$\bar{z} = re^{-i\theta}, r, \theta \in \mathbb{R}$
Conversion	$a = r\cos\theta,\ b = r\sin\theta$	$r = \sqrt{a^2 + b^2},\ \theta = \arctan\left(\frac{b}{a}\right)$
Re(z)	$a = \frac{1}{2}(z + \bar{z})$	$r\cos\theta = \frac{r}{2}(e^{i\theta} + e^{-i\theta})$
Im(z)	$b = \frac{1}{2}(z - \bar{z})$	$r\sin\theta = \frac{r}{2}(e^{i\theta} - e^{-i\theta})$

- In this course, we typically express the frequency variable s as a complex number $s = \sigma + i\omega$:

Euler's Formula

Definition

For any real number t, we define $e^{it} = \cos t + i \sin t$

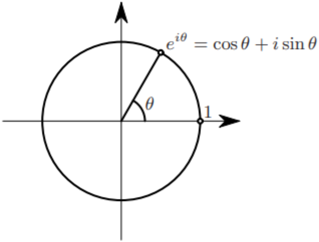

Figure: Euler's definition of $e^{i\theta}$.

Euler's formula: A famous example is $e^{\pi i} = \cos \pi + i \sin \pi = -1$, leading to

$$e^{\pi i} + 1 = 0.$$

This combines the five most basic quantities in mathematics $e, \pi, i, 1$ and 0.

Computation

- Addition and subtraction in Cartesian form tend to be simpler:

$$
\begin{aligned}
& z_1 = a + bi, \quad z_2 = c + di, \\
\Rightarrow \quad & z_1 + z_2 = a + c + (b + d)i, \\
\Rightarrow \quad & z_1 - z_2 = a - c + (b - d)i.
\end{aligned}
$$

- On the other hand, multiplication and division tend to be simpler in polar form:

$$
\begin{aligned}
& z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2}, \\
\Rightarrow \quad & z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}, \\
\Rightarrow \quad & \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.
\end{aligned}
$$

Multiplication and division and in the complex plane correspond to

- scaling the magnitude by the original magnitudes,
- and shifting phase by the sum or difference of the original phases.

Example

Example

Let $z_1 = -1 + \sqrt{3}i$ and $z_2 = \bar{z}_1$. Find polar forms for z_1 and z_2. Calculate $z_1 + z_2$, $z_1 z_2$, and $\frac{z_1}{z_2}$.

- In polar form, we have the magnitude is given by

$$|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2.$$

- The phase is given by

$$\theta = \arctan\left(\frac{\sqrt{3}}{-1}\right) = \frac{2\pi}{3}.$$

- Then, we have

$$z_1 = 2e^{\frac{2\pi}{3}i}, \quad z_2 = 2e^{\frac{-2\pi}{3}i}.$$

- In the Cartesian form, we have

$$z_2 = \bar{z}_1 = -1 - \sqrt{3}i, \qquad z_1 + z_2 = -2, \qquad z_1 - z_2 = 2\sqrt{3}i.$$

Example

- In the Cartesian form, we have

$$z_1 z_2 = (-1+\sqrt{3}i)(-1-\sqrt{3}i) = 1-\sqrt{3}i+\sqrt{3}i-3i^2 = 4,$$

$$\frac{z_1}{z_2} = \frac{-1+\sqrt{3}i}{-1-\sqrt{3}i}\cdot\frac{-1+\sqrt{3}i}{-1+\sqrt{3}i} = \frac{1-2\sqrt{3}i+3i^2}{4} = \frac{-1-\sqrt{3}i}{2}.$$

- In the polar form, we have

$$z_2 = \bar{z}_1 = 2e^{\frac{-2\pi}{3}i},$$

$$z_1+z_2 = 2\left(e^{\frac{2\pi}{3}i}+e^{\frac{-2\pi}{3}i}\right) = 2\left(2\cos\left(\frac{2\pi}{3}\right)\right) = -2,$$

$$z_1-z_2 = 2\left(e^{\frac{2\pi}{3}i}-e^{\frac{-2\pi}{3}i}\right) = 2\left(2\sin\left(\frac{2\pi}{3}\right)\right)i = 2\sqrt{3}i,$$

and

$$z_1\cdot z_2 = 2e^{\frac{2\pi}{3}i}\cdot 2e^{\frac{-2\pi}{3}i} = 4e^{\left(\frac{2\pi}{3}i-\frac{2\pi}{3}i\right)} = 4,$$

$$\frac{z_1}{z_2} = \frac{2e^{\frac{2\pi}{3}i}}{2e^{\frac{-2\pi}{3}i}} = e^{\frac{4\pi}{3}i}.$$

Outline

Complex functions

A **complex valued function**[1] on some interval $I = (a, b) \subseteq \mathbb{R}$ is $f : I \to \mathbb{C}$.

- Such a function can be written as in terms of its real and imaginary parts,

$$f(t) = u(t) + iv(t),$$

in which $u, v : I \to \mathbb{R}$ are two real valued functions.

A **complex valued function** of a complex variable is a function $f(z) : \mathbb{C} \to \mathbb{C}$.

- If $z = x + iy$, then $f(z)$ corresponds to a function

$$F(x, y) = u(x, y) + iv(x, y)$$

of the two real variables x and y.

- We can consider $f(z)$ is a function from $\mathbb{R}^2$ to $\mathbb{R}^2$.

[1] See `https://www.math.columbia.edu/~rf/complex2.pdf`.

Example

Example

1. $f(z) = z$ corresponds to $F(x, y) = x + iy \quad (u = x, v = y)$;
2. $f(z) = \bar{z}$ corresponds to $F(x, y) = x - iy \quad (u = x, v = -y)$;
3. $f(z) = \text{Re}(z)$ corresponds to $F(x, y) = x \quad (u = x, v = 0)$;
4. $f(z) = |z|$ corresponds to $F(x, y) = \sqrt{x^2 + y^2} \quad (u = \sqrt{x^2 + y^2}, v = 0)$;
5. $f(z) = z^2$ corresponds to $F(x, y) = (x^2 - y^2) + i(2xy)$
   $(u = x^2 - y^2, v = 2xy)$;
6. $f(z) = e^z$ corresponds to $F(x, y) = e^x \cos y + i(e^x \sin y)$
   $(u = e^x \cos y, v = e^x \sin y)$;
7. $f(z) = \frac{1}{z}$ corresponds to $F(x, y) = \dfrac{x}{x^2 + y^2} + i\dfrac{-y}{x^2 + y^2}$
   $(u = \dfrac{x}{x^2 + y^2}, v = \dfrac{-y}{x^2 + y^2})$;

Polynomials

A *polynomial* of a complex variable $z = x + iy$ is a function of the form

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0,$$

where $a_i, i = 0, 1, \ldots, n$ are complex numbers.

- We focus on polynomials with real coefficient ($a_i \in \mathbb{R}, i = 1, \ldots, n$).
- The real and imaginary parts of a polynomial $P(z)$ are polynomials in x, y:

$$P_1(z) = z^2 = (x^2 - y^2) + i(2xy),$$
$$P_2(z) = (1+i)z^2 - 3iz = (x^2 - y^2 - 2xy + 3y) + (x^2 - y^2 + 2xy - 3x)i.$$

- In the polar form $z = re^{i\theta}$, we have

$$z^n = r^n e^{in\theta}, \qquad (\bar{z})^n = r^n e^{-in\theta}.$$

- Thus, we have $\overline{(z^n)} = (\bar{z})^n$.
- In general, for any polynomial $p(z)$ with real coefficients, we have

$$\overline{p(z)} = p(\bar{z}), \qquad \overline{p(i\omega)} = p(-i\omega)$$

Fundamental Theorem of Algebra

Fundamental Theorem of Algebra (first proved by Gauss in 1799): if $P(z)$ is a non-constant polynomial, then $P(z)$ has a complex root. In other words, there exists a complex number c such that $P(c) = 0$.

- If $P(z)$ is a polynomial of degree $n > 0$, then $P(z)$ can be factorized into linear factors:
$$P(z) = a(z - \lambda_1) \cdots (z - \lambda_n),$$
for complex numbers a and $\lambda_1, \ldots, \lambda_n$.

Every non-constant polynomial $P(z)$ with real coefficients can be factorized into (real) polynomials of degree one or two.

- In other words, the roots of polynomial $P(z)$ with real coefficients come with pairs $\lambda_i = x + yi$ and $\lambda_{i+1} = x - yi$.

Rational functions

A rational function $G(z)$ is a quotient of two polynomials

$$G(z) = \frac{P(z)}{Q(z)},$$

where $P(z)$ and $Q(z)$ are polynomials and $Q(z)$ is not identically zero.

Example

Here are some examples

$$G_1(z) = \frac{1}{z},$$

$$G_2(z) = \frac{1}{z+1},$$

$$G_3(z) = \frac{z+1}{z^2+z+1}.$$

Outline

Laplace transform

Definition

The Laplace transform of a function $f(t)$ is defined by the integral

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt,$$

for those complex variable s where the integral converges.

The Laplace transform[2] takes a function of time and transforms it to a function of a complex variable s.

- Because the transform is invertible, no information is lost
- It is reasonable to think of a function $f(t)$ and its Laplace transform $F(s)$ as two views of the same phenomenon.
- Each view has its uses and some features of the phenomenon are easier to understand in one view or the other.

[2] See `https://math.mit.edu/~jorloff/18.04/notes/topic12.pdf`.

Example 1

Example

Calculate the Laplace transform of the step function

$$f(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0. \end{cases}$$

Give the region in the complex s-plane where the integral converges. We have

$$\begin{aligned} F(s) = \mathcal{L}(f(t)) &= \int_0^\infty e^{-st} f(t) dt \\ &= \int_0^\infty e^{-st} dt \\ &= \left. \frac{e^{-st}}{-s} \right|_0^\infty \\ &= \frac{1}{s}, \end{aligned}$$

if $\text{Re}(s) > 0$, otherwise it is undefined.

Example 2

Example

Calculate the Laplace transform of the shifted delta function

$$\delta(t-a) = \begin{cases} \infty & t = a \\ 0 & \text{otherwise} \end{cases}, \qquad \int_{-\infty}^{\infty} \delta(t-a)dt = 1.$$

By definition, we have

$$\begin{aligned} F(s) = \mathcal{L}(f(t)) &= \int_0^\infty e^{-st}\delta(t-a)dt \\ &= \int_0^\infty e^{-sa}\delta(t-a)dt \\ &= e^{-sa}\int_0^\infty \delta(t-a)dt \\ &= e^{-sa}. \end{aligned}$$

In particular, when $a = 0$, we have

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st}\delta(t)dt = 1.$$

Example 3

Example

Calculate the Laplace transform of the exponential function

$$f(t) = e^{at}.$$

- Given the region in the complex s-plane where the integral converges.
- We have

$$\begin{aligned} F(s) = \mathcal{L}(f(t)) &= \int_0^\infty e^{-st} e^{at} dt \\ &= \int_0^\infty e^{(a-s)t} dt \\ &= \frac{e^{(a-s)t}}{a-s} \Big|_0^\infty \\ &= \frac{1}{s-a}, \end{aligned}$$

if $\mathrm{Re}(s) > \mathrm{Re}(a)$, otherwise, it is undefined.

Example 4

Example

Compute the Laplace transform of the cosine function

$$f(t) = \cos(\omega t).$$

- We use the formula
$$\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}.$$
- So we have
$$\begin{aligned} F(s) = \mathcal{L}(f(t)) &= \int_0^\infty e^{-st} \left(\frac{e^{i\omega t} + e^{-i\omega t}}{2} \right) dt \\ &= \frac{1}{2} \left(\frac{1}{s - i\omega} + \frac{1}{s + i\omega} \right) \\ &= \frac{s}{s^2 + \omega^2}. \end{aligned}$$