# ECE 171A: Linear Control System Theory
# Discussion 6: Bode plot - Examples

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# Outline

Bode plot

First-order and second-order examples

High-order examples

The Routh–Hurwitz Criterion: Examples

# Outline

# Bode plot — Blocks

A polynomial can be written as a product of terms of the type

$$k, \qquad s, \qquad s+a, \qquad s^2 + 2\zeta\omega_0 s + \omega_0^2$$

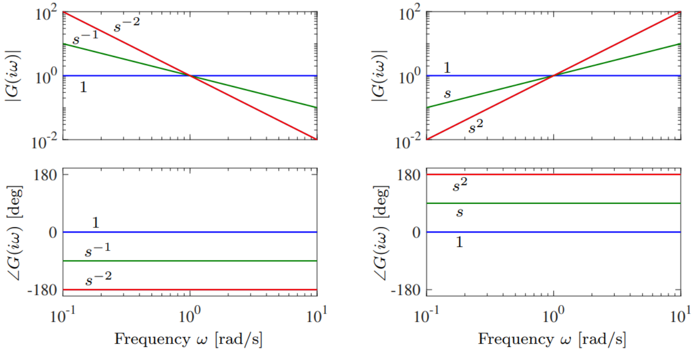
- Sketch Bode diagrams for these terms;
- Complex systems: add the gains and phases of the individual terms

**Case 1**: $G(s) = s^k$ — Two special cases: $k = 1$, a differentiator; $k = -1$, an integrator

$$\log|G(s)| = k \times \log\omega, \qquad \angle G(i\omega) = k \times 90^\circ$$

- The gain curve is a straight line with slope $k$, and the phase curve is a constant at $k \times 90^\circ$
- The case when $k = 1$ corresponds to a differentiator and has slope 1 with phase $90^\circ$
- The case when $k = -1$ corresponds to an integrator and has slope $-1$ with phase $-90^\circ$

# Case 1: $G(s) = s^k$

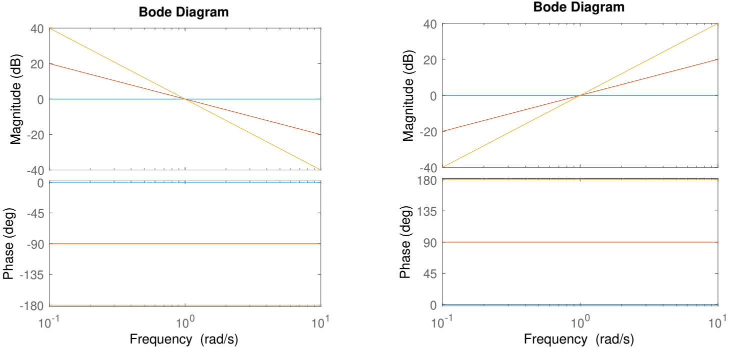



(a) $s^k$, $k < 0$ (b) $s^k$, $k > 0$

Figure: Bode plots of the transfer functions $G(s) = s^k$ for $k = -2, -1, 0, 1, 2$. On a log-log scale, the gain curve is a straight line with slope $k$. The phase curves for the transfer functions are constants, with phase equal to $k \times 90°$.

# Case 1: $G(s) = s^k$

```
G0 = tf([1],[1]); % create a transfer function
G1 = tf([1 0],[1]); % create a transfer function
W = {0.1,10}; bode(G0,G1,W); % Bode plot
```




(a) $s^k, k < 0$ (b) $s^k, k > 0$

Figure: Bode plots of the transfer functions $G(s) = s^k$ for $k = -2, -1, 0, 1, 2$ — from Matlab

# Outline

# Case 2: first-order system

Consider the transfer function of a first-order system

$$G(s) = \frac{a}{s+a}, \qquad a > 0.$$

- We have

$$|G(s)| = \frac{|a|}{|s+a|}, \qquad \angle G(s) = \angle a - \angle(s+a).$$

- The gain curve is

$$|G(i\omega)| = \log a - \frac{1}{2}\log(\omega^2 + a^2) \approx \begin{cases} 0, & \text{if } \omega < a \\ \log a - \log \omega, & \text{if } \omega > a \end{cases}$$

- The phase curve is

$$\angle G(i\omega) = -\frac{180}{\pi}\arctan\frac{\omega}{a} \approx \begin{cases} 0, & \text{if } \omega < \frac{a}{10} \\ -45 - 45(\log\omega - \log a), & \text{if } a/10 < \omega < 10a \\ -90, & \text{if } \omega > 10a \end{cases}$$

# Case 2: first-order system

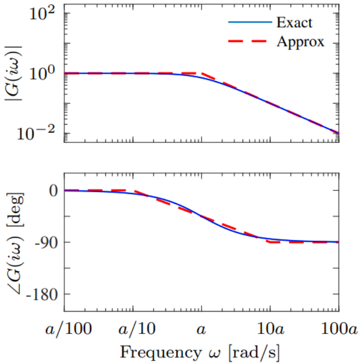



Figure: Bode plot of the first-order system $G(s) = a/(s + a)$, which can be approximated by asymptotic curves (dashed) in both the gain and the frequency, with the breakpoint in the gain curve at $\omega = a$ and the phase decreasing by $90^\circ$ over a factor of 100 in frequency.

A first-order system behaves like a constant for low frequencies and like an integrator for high frequencies.

## Case 3: second-order system

Consider the transfer function of a first-order system

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}, \qquad 0 < \zeta < 1.$$

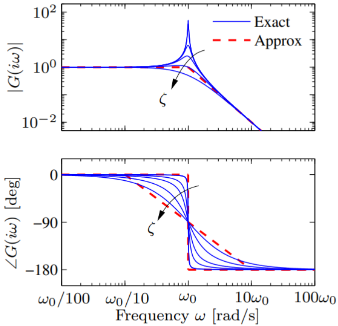
- The gain curve is

$$\begin{aligned} |G(i\omega)| &= 2\log\omega_0 - \frac{1}{2}\log(\omega^4 + 2\omega_0^2\omega^2(2\zeta^2 - 1) + \omega_0^4) \\ &\approx \begin{cases} 0, & \text{if } \omega \ll \omega_0 \\ 2\log\omega_0 - 2\log\omega, & \text{if } \omega \gg \omega_0 \end{cases} \end{aligned}$$

- The largest gain $Q = \max_\omega |G(i\omega)| \approx 1/(2\zeta)$, called the Q-value, is obtained for $\omega \approx \omega_0$ – **Resonant frequency**
- The phase curve is

$$\angle G(i\omega) = -\frac{180}{\pi}\arctan\frac{2\zeta\omega_0\omega}{\omega_0^2 - \omega} \approx \begin{cases} 0, & \text{if } \omega \ll \omega_0 \\ -180, & \text{if } \omega \gg \omega_0 \end{cases}$$

# Case 3: Second-order system




(b) Second-order system

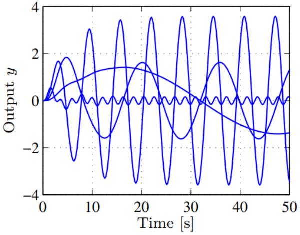
Figure: Bode plot of the second-order system $G(s) = \omega_0^2/(s^2 + 2\zeta\omega_0 s + \omega_0^2)$, which has a peak at frequency $\omega_0$ and then a slope of $-2$ beyond the peak; the phase decreases from $0^\circ$ to $-180^\circ$. The height of the peak and the rate of change of phase depending on the damping ratio $\zeta$ ($\zeta = 0.02, 0.1, 0.2, 0.5$, and $1.0$ shown).

The asymptotic approximation is poor near $\omega = \omega_0$ and that the Bode plot depends strongly on $\zeta$ near this frequency.

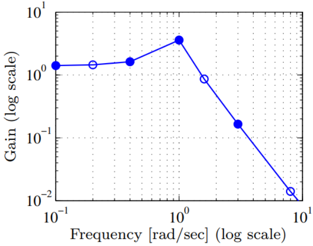
# Determine Transfer function experimentally

Model a given application by measuring the frequency response

- Apply a sinusoidal signal at a fixed frequency.
- Measure the amplitude ratio and phase lag when steady state is reached.
- The complete frequency response is obtained by sweeping over a range of frequencies.




(a) Time domain simulations




(b) Frequency response

Figure: A frequency response (gain only) computed by measuring the response of individual sinusoids.

# Outline

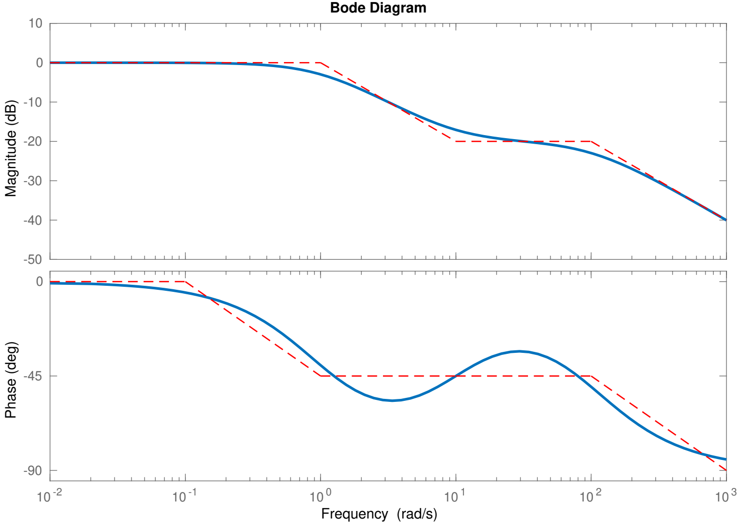
# High-order Example I

## Example

Draw a Bode plot for $G(s) = 10\frac{s+10}{(s+1)(s+100)}$

- Step 1: find break points (poles and zeros): 1, 10, 100.
- Step 2: Calculate $|G(i0)|$ and $\angle G(i0)$ to determine the starting points
- Step 3: Sketch the bode plot by the rules
  - **Magnitude increases with a zero**: if the zero is a first-order real zero, the slop is $+1$; if the zero is a second-order zero (or complex zero), the slop is $+2$
  - **Magnitude decreases with a pole**: If there pole is a first-order real pole, the slop is $-1$; if the pole is a second-order pole (or complex pole), the slop is $-2$
  - **Phases changes** by $+90$ with a first order real zero; $+180$ with a second order zero (or complex zero). The change starts around $a/10$ and ends around $10a$.
  - **Phases changes** by $-90$ with a first order real pole; $-180$ with a second order pole (or complex pole). Similarly, the change starts around $a/10$ and ends around $10a$.

# High-order Example I



# Another example II

## Example

Draw bode plot for

$$G(s) = \frac{k(s+b)}{(s+a)(s^2+2\zeta\omega_0 s+\omega_0^2)}, \qquad a \ll b \ll \omega_0.$$

- **Gain curve:**
  - Begin with low frequency $G(0) = \frac{kb}{a\omega_0^2}$.
  - Reach $\omega = a$, the effect of the pole begins and the gain decreases with slope $-1$
  - At $\omega = b$, the zero comes into play and we increase the slope by 1, leaving the asymptote with net slope 0.
  - This slope is used until the effect of the second-order pole is seen at $\omega = \omega_0$, at which point the asymptote changes to slope $-2$.
- **Phase curve:**
  - The approximation process is similar, but slightly more complicated since the effect of the phase stretches out much further.

# Another example II

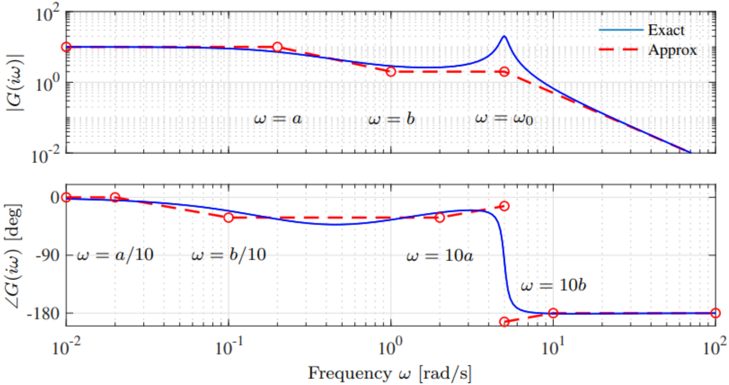



**Figure 9.15:** Asymptotic approximation to a Bode plot. The solid curve is the Bode plot for the transfer function $G(s) = k(s+b)/(s+a)(s^2+2\zeta\omega_0 s+\omega_0^2)$, where $a \ll b \ll \omega_0$. Each segment in the gain and phase curves represents a separate portion of the approximation, where either a pole or a zero begins to have effect. Each segment of the approximation is a straight line between these points at a slope given by the rules for computing the effects of poles and zeros.

# Outline

# Stability

**Theorem (Stability of a linear system (Lyapunov sense))**

*The system* $\dot{x} = Ax$ *is*

- **asymptotically stable** *if and only if all eigenvalues of* $A$ *have a strictly negative real part, i.e.,* $\mathrm{Re}(\lambda_i) < 0$
- **unstable** *if any eigenvalues* $A$ *has a strictly positive real part.*

Consider an LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du \end{aligned} \qquad \Longleftrightarrow \qquad G(s) = C(sI - A)^{-1}B + D$$

**Poles (eigenvalues) of the matrix** $A$ **= Poles of the transfer function** $G(s)$

- A system is **bounded-input bounded-output (BIBO)** stable if every bounded input $u(t)$ leads to a bounded output $y(t)$.
- BIBO stable: if all poles of $G(s)$ are in the open left half-plane in the $s$ domain (i.e., having negative real parts).

# Routh Table

- $a(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0$

| | | | | | |
|---|---|---|---|---|---|
| $s^n$ | $a_n$ | $a_{n-2}$ | $a_{n-4}$ | $\cdots$ | $a_0$ |
| $s^{n-1}$ | $a_{n-1}$ | $a_{n-3}$ | $a_{n-5}$ | $\cdots$ | $0$ |
| $s^{n-2}$ | $b_{n-1} = -\frac{\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}}$ | $b_{n-3} = -\frac{\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{a_{n-1}}$ | $b_{n-5}$ | $\cdots$ | $0$ |
| $s^{n-3}$ | $c_{n-1} = -\frac{\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}}{b_{n-1}}$ | $c_{n-3} = -\frac{\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_{n-1} & b_{n-5} \end{vmatrix}}{b_{n-1}}$ | $c_{n-5}$ | $\cdots$ | $0$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ldots$ | $\vdots$ |
| $s^0$ | $a_0$ | $0$ | $0$ | $\cdots$ | $0$ |

- Any row can be multiplied by a positive constant without changing the result

# Example: Higher-order System

## Example

Consider the characteristic polynomial of a fifth-order system:

$$a(s) = s^5 + s^4 + 10s^3 + 72s^2 + 152s + 240$$

- The Routh table is:

| | | | |
|---|---|---|---|
| $s^5$ | $1$ | $10$ | $152$ |
| $s^4$ | $1$ | $72$ | $240$ |
| $s^3$ | $-62$ | $-88$ | $0$ |
| $s^2$ | $70.6$ | $240$ | $0$ |
| $s^1$ | $122.6$ | $0$ | $0$ |
| $s^0$ | $240$ | $0$ | $0$ |

- Since there are two sign changes in the first column, there are two roots in the right half-plane and the system is **unstable**
- The roots of $a(s)$ are:

$$a(s) = (s+3)(s+1 \pm j\sqrt{3})(s-2 \pm j4)$$

# Example: Special Case

## Example

Consider the polynomial:

$$a(s) = s^4 + s^3 + 2s^2 + 2s + 3$$

- The Routh table is:

| | | | |
|---|---|---|---|
| $s^4$ | 1 | 2 | 3 |
| $s^3$ | 1 | 2 | 0 |
| $s^2$ | $\not{0}$ → $\epsilon$ | 3 | 0 |
| $s^1$ | $2 - \frac{3}{\epsilon}$ | 0 | 0 |
| $s^0$ | 3 | 0 | 0 |

- For $0 < \epsilon \ll 1$, we see that $2 - \frac{3}{\epsilon} < 0$
- Since there are two sign changes in the first column, there are two roots in the right half-plane and the system is **unstable**
- The roots are $p_{1,2} = 0.4057 \pm 1.2928i, \qquad p_{3,4} = -0.9057 \pm 0.9020i$

# Example: Special Case 2

## Example

Consider the polynomial:

$$a(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

- The Routh table is:

| | | | |
|---|---|---|---|
| $s^5$ | $1$ | $2$ | $11$ |
| $s^4$ | $2$ | $4$ | $10$ |
| $s^3$ | $\not{0} \to \epsilon$ | $6$ | $0$ |
| $s^2$ | $c_4 = \frac{1}{\epsilon}(4\epsilon - 12)$ | $10$ | $0$ |
| $s^1$ | $d_4 = \frac{1}{c_4}(6c_4 - 10\epsilon)$ | $0$ | $0$ |
| $s^0$ | $10$ | $0$ | $0$ |

- For $0 < \epsilon \ll 1$, we see that $c_4 < 0$ and $d_4 > 0$
- Since there are two sign changes in the first column, there are two roots in the right half-plane and the system is **unstable**
- The roots are
  $\lambda_{1,2} = 0.8950 \pm 1.4561i, \lambda_{3,4} = -1.2407 \pm 1.0375i, \lambda_5 = -1.3087.$