ECE 171A: Linear Control System Theory
Discussion 7: Nyquist plot - Review & Examples

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Outline

Nyquist stability criterion

Nyquist plot - examples

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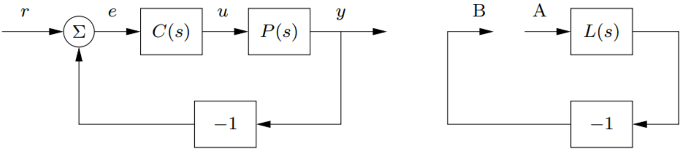
Stability of feedback systems



(a) Closed loop system

(b) Open loop system

- **Lyapunov stability** — eigenvalue test of the closed-loop matrix; e.g.,

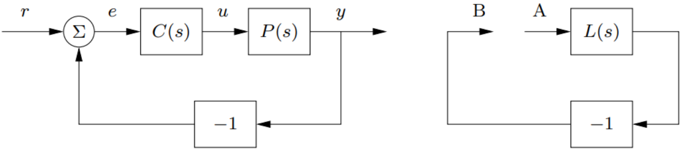
$$\begin{aligned} \text{Dynamics} \rightarrow \quad & \dot{x} = Ax + Bu, \\ \text{Feedback controller} \rightarrow \quad & u = -Kx \end{aligned} \quad \Rightarrow \quad \dot{x} = (A - BK)x.$$

- **Poles or The Routh–Hurwitz Criterion**;

$$\begin{cases} P(s) & = \dfrac{n_{\mathrm{p}}(s)}{d_{\mathrm{p}}(s)} \\ C(s) & = \dfrac{n_{\mathrm{c}}(s)}{d_{\mathrm{c}}(s)} \end{cases} \quad \Rightarrow \quad G_{yr}(s) = \frac{PC}{1+PC} = \frac{n_{\mathrm{p}}(s) n_{\mathrm{c}}(s)}{d_{\mathrm{p}}(s) d_{\mathrm{c}}(s) + n_{\mathrm{p}}(s) n_{\mathrm{c}}(s)}$$

They are **straightforward but give little guidance** for design: it is not easy to tell how the controller should be modified to make an unstable system stable.

Nyquist's idea



(a) Closed loop system (b) Open loop system

- Nyquist's idea was to first investigate conditions under which oscillations can occur in a feedback loop.
- The **Loop transfer function**:

$$L(s) = P(s)C(s).$$

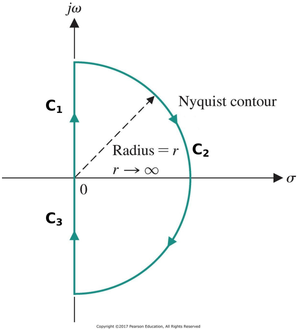
- Assume that a sinusoid of frequency ω_0 is injected at point A. In steady state, the signal at point B will also be a sinusoid with the frequency ω_0.

Very intuitive idea: It seems reasonable that an oscillation can be maintained if the signal at B has the same amplitude and phase as the injected signal!

Nyquist contour

The (standard or simplest) Nyquist contour, also known as "Nyquist D contour" ($\Gamma \subset \mathbb{C}$), is made up of three parts:

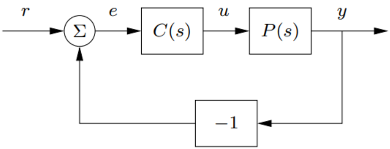
- **Contour** C_1: points $s = i\omega$ on the positive imaginary axis, as ω ranges from 0 to ∞
- **Contour** C_2: points $s = Re^{i\theta}$ on a semi-circle as $R \to \infty$ and θ ranges from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$
- **Contour** C_3: points $s = i\omega$ on the negative imaginary axis, as ω ranges from $-\infty$ to 0



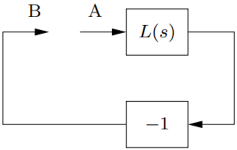
The image of $L(s)$ when s traverses Γ gives a closed curve in the complex plane and is referred to as the **Nyquist plot** for $L(s)$.

- **Nyquist's stability criterion** utilizes **contours** in the complex plane to relate the **locations** of the open-loop and closed-loop poles.

Simplified Nyquist Criterion



(a) Closed loop system



(b) Open loop system

Theorem (Simplified Nyquist Criterion)

Let $L(s)$ be the loop transfer function for a negative feedback system, and assume that L has no poles in the closed right half-plane ($\mathrm{Re}(s) \geq 0$) except possibly at the origin. Then the closed loop system

$$G_{\mathrm{cl}}(s) = \frac{L(s)}{1 + L(s)}$$

is stable if and only if the image of $L(s)$ along the closed contour Γ has no net encirclements of the critical point $s = -1 + i0$.

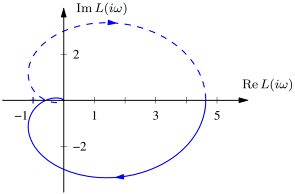
Winding number

The following conceptual procedure can be used to determine that there are no net encirclements.

- Step 1: Fix a pin at the critical point $s = -1$, orthogonal to the plane.
- Step 2: Attach a string with one end at the critical point and the other on the Nyquist plot.
- Step 3: Let the end of the string attached to the Nyquist curve traverse the whole curve.

There are no net encirclements if the string does not wind up on the pin when the curve is encircled.

- The number of net encirclements is called the **winding number**.



Nyquist plot for $L(s) = \frac{1}{(s+a)^3}$ with $a = 0.6$

- Closed-loop system

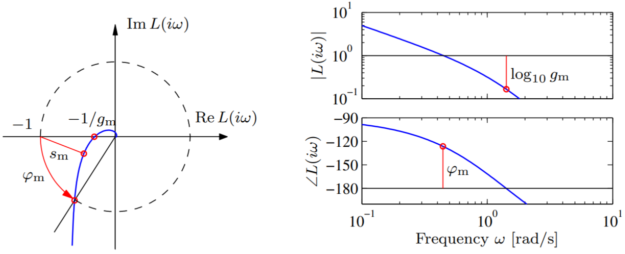
$$G_{\text{cl}}(s) = \frac{L(s)}{1 + L(s)} = \frac{1}{(s+0.6)^3 + 1}, \qquad \lambda_1 = -1.6000, \lambda_{2,3} = -0.1 \pm 0.8660i$$

Nyquist's Stability Criterion

Theorem (Nyquist's Stability Criterion)

Consider a unity feedback control system with open-loop transfer function $L(s)$. Let Γ be a Nyquist contour. The system is stable if and only if **the number of counterclockwise encirclements** *of $-1 + i0$ by the Nyquist plot $L(\Gamma)$ is equal to* **the number of poles** *of $L(s)$ inside Γ.*

Classical robustness measures: stability margin, phase margin, gain margin



(a) Nyquist plot (b) Bode plot

Outline

Example 1: a third-order system

Draw a Nyquist plot for $L(s) = \frac{1}{(s+a)^3}$.

- **Counter** C_1: $s = i\omega$ with ω from 0 to ∞

$$L(i0) = \frac{1}{a^3}\angle 0^\circ, \qquad L(i\infty) = 0\angle - 270^\circ$$

- for $0 < \omega < \infty$

$$L(i\omega) = \frac{1}{(i\omega + a)^3} = \frac{(a - i\omega)^3}{(a^2 + \omega^2)^3} = \frac{a^3 - 3a\omega^2}{(a^2 + \omega^2)^3} + i\frac{\omega^3 - 3a^2\omega}{(a^2 + \omega^2)^3}$$

- **Counter** C_2: $s = Re^{i\theta}$ for $R \to \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$.

$$L(Re^{i\theta}) = \frac{1}{(Re^{i\theta} + a)^3} \to 0$$

- **Counter** C_3: $s = i\omega$ with $\omega \in (-\infty, 0)$

$$L(-i\omega) = L(\bar{i}\omega) = \overline{L(i\omega)}$$

which is a *reflection* (complex conjugate) of $L(C_1)$ about the real axis.

Example 1: a third-order system

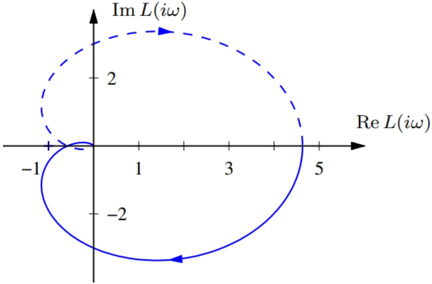


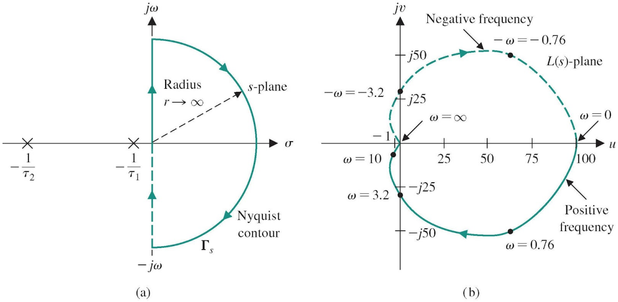
Figure 10.5: Nyquist plot for a third-order transfer function $L(s)$. The Nyquist plot consists of a trace of the loop transfer function $L(s) = 1/(s+a)^3$ with $a = 0.6$. The solid line represents the portion of the transfer function along the positive imaginary axis, and the dashed line the negative imaginary axis. The outer arc of the Nyquist contour Γ maps to the origin.

Example 2: a second-order system

Draw a Nyquist plot for

$$L(s) = \frac{100}{(1+s)(1+s/10)}.$$

- **Contour** C_1: $L(i0) = 100\angle 0^\circ$, $L(i\infty) = 0\angle{-180^\circ}$
- **Contour** C_2: $\lim_{R\to\infty} L(Re^{i\theta}) = 0$



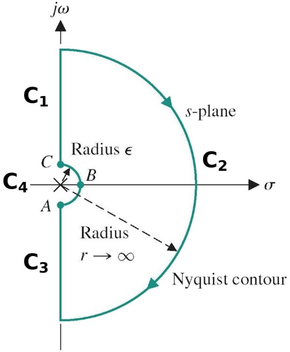
Pole/Zero on the Imaginary Axis

- When the loop transfer function has poles on the imaginary axis, the gain is infinite at the poles.
- The Nyquist contour needs to be modified to take a small detour around such poles or zeros
- So, we add another part: **Contour** C_4
  - plot $L(\epsilon e^{i\theta})$ for $\epsilon \to 0$ and

$$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

  - substitute $s = \epsilon e^{i\theta}$ into $L(s)$ and examine what happens as

$$\epsilon \to 0$$

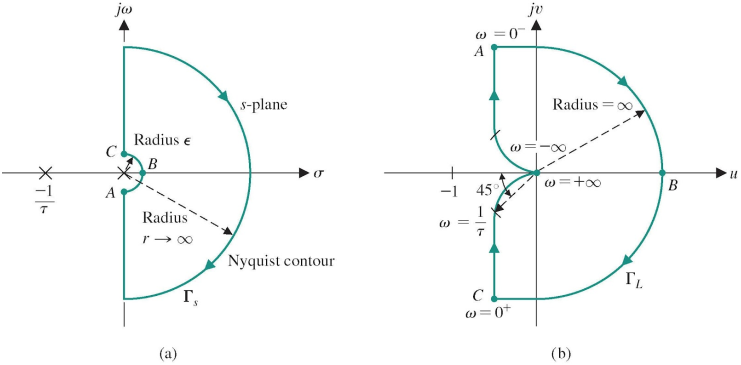


Example 3

Draw a Nyquist plot for a loop transfer system:

$$L(s) = \frac{\kappa}{s(1+\tau s)}$$

- Since there is a pole at the origin, we need to use a modified Nyquist contour



Example 3

- **Contour** C_4 with $s = \epsilon e^{i\theta}$ for $\epsilon \to 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$:

$$\lim_{\epsilon \to 0} L(\epsilon e^{i\theta}) = \lim_{\epsilon \to 0} \frac{\kappa}{\epsilon e^{i\theta}} = \lim_{\epsilon \to 0} \frac{\kappa}{\epsilon} e^{-i\theta} = \infty \angle -\theta$$

  - The phase of $L(s)$ changes from $\dfrac{\pi}{2}$ at $\omega = 0^-$ to $-\dfrac{\pi}{2}$ at $\omega = 0^+$
- **Contour** C_1 with $\omega \in (0, \infty)$:

$$\begin{aligned} L(i0^+) &= \infty \angle -90^\circ \\ L(i\infty) &= \lim_{\omega \to \infty} \frac{\kappa}{i\omega(1 + i\omega\tau)} = \lim_{\omega \to \infty} \left| \frac{\kappa}{\tau \omega^2} \right| \angle -\pi/2 - \tan^{-1}(\omega\tau) \\ &= 0 \angle -180^\circ \end{aligned}$$

- **Contour** C_2 with $s = re^{i\theta}$ for $r \to \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$:

$$\lim_{r \to \infty} L(re^{i\theta}) = \lim_{r \to \infty} \left| \frac{\kappa}{\tau r^2} \right| e^{-2i\theta} = 0 \angle -2\theta$$

  - The phase of $L(s)$ changes from $-\pi$ at $\omega = \infty$ to π at $\omega = -\infty$
- **Contour** C_3 with $\omega \in (-\infty, 0)$:
  - $L(C_3)$ is a **reflection** of $L(C_1)$ about the real axis

Summary - Nyquist contour

- *Open-loop transfer function*: $L(s) = P(s)C(s)$
- *Close-loop transfer function*

$$G_{\mathrm{yr}} = \frac{L(s)}{1 + L(s)}$$

Nyquist Contour is a D-shape curve in the complex domain, avoiding all the poles of $L(s)$ on the imaginary axis.

- Only poles on the imaginary axis needs to be avoided.
- The default orientation of traveling along the contour is clockwise.
- The semi-circle centered at pole p on the imaginary axis, rotating in counter clockwise direction, is represented by $p + Re^{i\theta},\ \theta \sim -\frac{\pi}{2} \to \frac{\pi}{2}$.
- The big semi-circle of the contour, rotating in clockwise direction, is represented by $Re^{i\theta},\ \theta : +\frac{\pi}{2} \to -\frac{\pi}{2}, R \to \infty$

Summary - Nyquist plots

The **Nyquist Plot** is the image of the **Nyquist Contour** after going through the function $L(s)$. Nyquist Contour $D \Rightarrow$ Nyquist Plot$L(D)$.

- Start with the expression of $L(s)$ when s is on the imaginary axis $s = i\omega$.
- When drawing the plot, it is helpful to first think about how $\mathrm{Re}(L(s))$ will change, then think about how $\mathrm{Im}(L(s))$ will change.
- In many cases, $|L(s)| \to 0$ when $|s| \to \infty$. Many Nyquist plots stuck at 0 as you travel along the big semi-circle of the Nyquist Contour.
- The part of the Nyquist Plot corresponding to the lower half of imaginary axis in the Nyquist Contour is **symmetrical** (reflection) to the other half.
- Cautions with using MATLAB
  - MATLAB doesn't generate the portion of plot for corresponding to the poles on imaginary axis
  - These must be drawn in by hand (get the orientation right!)

Theorem (Nyquist stability theorem)

$1 + L(s)$ *has* $Z = N + P$ *zeros in the right half plane.*

Example 4

$$L(s) = \frac{1}{s+1}$$

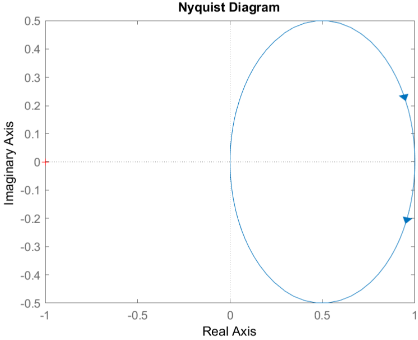


Figure: Nyquist plot for $L(s) = \frac{1}{s+1}$

$$Z = N + P = 0$$

Then,

$$G_{\text{yr}} = \frac{L(s)}{1 + L(s)}$$
$$= \frac{1}{s+2}$$

Example 5

$$L(s) = \frac{1}{(s+1)^2}$$

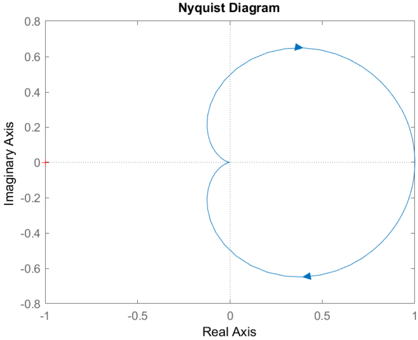


Figure: Nyquist plot for $L(s) = \frac{1}{(s+1)^2}$

$$Z = N + P = 0$$

Then,

$$G_{\text{yr}} = \frac{L(s)}{1 + L(s)} = \frac{1}{s^2 + 2s + 2}$$

Closed-loop poles

$$p_{1,2} = -1 \pm 1i$$

Example 6

$$L(s) = \frac{1}{s(s+1)}$$

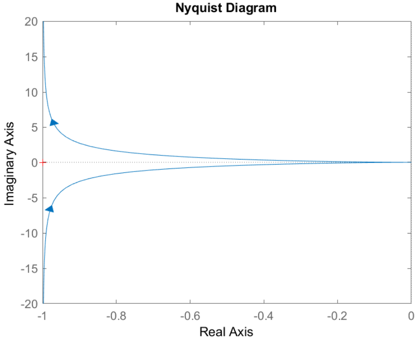


Figure: Nyquist plot for $L(s) = \frac{1}{s(s+1)}$

$$Z = N + P = 0$$

Then,

$$G_{\text{yr}} = \frac{L(s)}{1 + L(s)} = \frac{1}{s^2 + s + 1}$$

Closed-loop poles

$$p_{1,2} = -0.5 \pm 0.866i$$

Example 7

$$L(s) = \frac{1}{s(s+1)(s+0.5)}$$

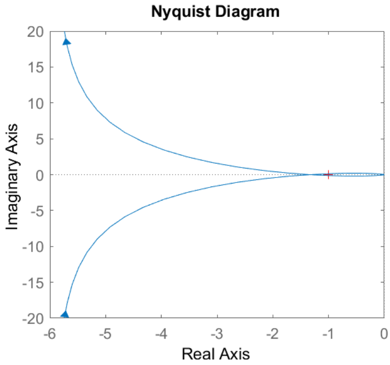


Figure: Nyquist plot for $L(s) = \frac{1}{s(s+1)(s+0.5)}$

$$Z = N + P = 2$$

Then,

$$G_{\text{yr}} = \frac{L(s)}{1+L(s)}$$
$$= \frac{1}{s^3 + 1.5s^2 + 0.5s + 1}$$

Closed-loop poles

$$p_{1,2} = 0.0416 \pm 0.7937i$$
$$p_3 = -1.5832$$