ECE 171A: Linear Control System Theory
Lecture 10: Input/output system responses (I)

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Reading materials: Ch 6.2, 6.3

Outline

Linear properties of LTI systems

Initial response - matrix exponential

Step, impulse and frequency responses

Summary

Outline

Linear time-invariant (LTI) systems

An LTI system is in the form of

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- $x \in \mathbb{R}^n$: state; $y \in \mathbb{R}^p$: output; $u \in \mathbb{R}^m$: input;
- $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$ are constant matrices.

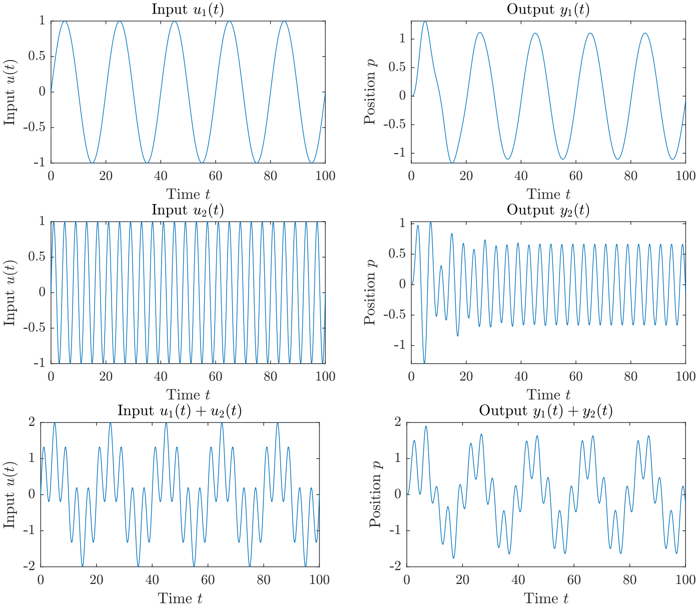
Two important features

- **Linear**: $f(x,u) = Ax + Bu$ and $h(x,u) = Cx + Du$ are linear functions
- **Time-invariant**: A, B, C, D do not change over time.

The output $y(t)$ has very nice **linear properties** in the following sense

- Zero initial state $x(0) = 0$: *the output* $y(t)$ *is linear in input* $u(t)$;
- Zero input $u(t) = 0$: *the output* $y(t)$ *is linear in initial states* $x(0)$.

Case 1: Zero initial state $x(0) = 0$



Case 1: Zero initial state $x(0) = 0$

Fact 1: Zero initial state $x(0) = 0$. The output $y(t)$ is linear in input

$$\begin{cases} u_1(t) \to y_1(t) \\ u_2(t) \to y_2(t) \end{cases} \implies \quad \alpha u_1(t) + \beta u_2(t) \to \alpha y_1(t) + \beta y_2(t)$$

Proof: Denote the state trajectories for $u_1(t)$ as $x_1(t)$, and for $u_2(t)$ as $x_2(t)$.

- By definition, we have

$$\dot{x}_1(t) = Ax_1(t) + Bu_1(t), \qquad \dot{x}_2(t) = Ax_2(t) + Bu_2(t),$$

- Let $x = \alpha x_1 + \beta x_2$, and verify $x(t)$ is a solution if $u(t) = \alpha u_1(t) + \beta u_2(t)$
- **Step 1**: the initial condition $x(0) = \alpha x_1(0) + \beta x_2(0) = 0$ is satisfied.
- **Step 2**: it is easy to verify that

$$\begin{aligned} \dot{x} = \alpha \dot{x}_1 + \beta \dot{x}_2 &= \alpha(Ax_1 + Bu_1) + \beta(Ax_2 + Bu_2) \\ &= A(\alpha x_1 + \beta x_2) + B(\alpha u_1(t) + \beta u_2(t)) \\ &= Ax + Bu. \end{aligned}$$

Case 1: Zero initial state $x(0) = 0$

Fact 1: Zero initial state $x(0) = 0$. The output $y(t)$ is linear in input

$$\begin{cases} u_1(t) \to y_1(t) \\ u_2(t) \to y_2(t) \end{cases} \implies \alpha u_1(t) + \beta u_2(t) \to \alpha y_1(t) + \beta y_2(t)$$

Proof:

- Finally, we have

$$\begin{aligned} y(t) &= Cx(t) + Du(t) \\ &= C(\alpha x_1 + \beta x_2) + D(\alpha u_1 + \beta u_2) \\ &= \alpha(Cx_1 + Du_1) + \beta(Cx_2 + Du_2) \\ &= \alpha y_1(t) + \beta y_2(t). \end{aligned}$$

- Therefore, we have proved that

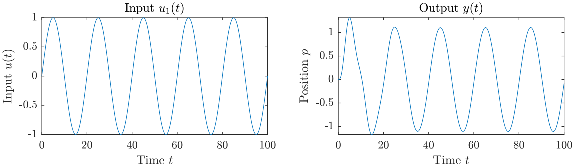
$$\alpha u_1(t) + \beta u_2(t) \to \alpha y_1(t) + \beta y_2(t)$$

Immediate corollary

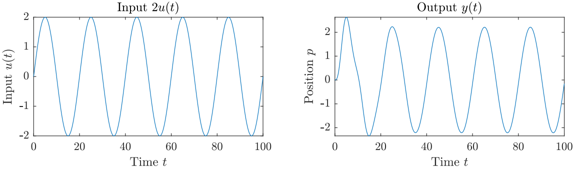
Given any LTI system with zero initial condition, *its system response scales with the input amplitude*

- If the input signal becomes twice as strong, then the strength of output will also double.
- This allows us to use ratios and percentages in *step* or *frequency* response. These are independent of input amplitudes.

Case 1:



Case 2:



Case 2: Zero input $u(t) = 0$

Fact 2: Zero input $u(t) = 0$. The output $y(t)$ is linear in the initial conditions

$$\begin{cases} x(0) \to y_1(t) \\ \hat{x}(0) \to y_2(t) \end{cases} \implies \quad \alpha x(0) + \beta \hat{x}(0) \to \alpha y_1(t) + \beta y_2(t)$$

Proof: Denote the state trajectories for $x(0)$ as $x_1(t)$, and for $\hat{x}(0)$ as $x_2(t)$.

- By definition, we have

$$\dot{x}_1(t) = Ax_1(t), \qquad \dot{x}_2(t) = Ax_2(t),$$

- Let $x = \alpha x_1 + \beta x_2$, and verify $x(t)$ is a solution.
- Step 1: the initial condition $x(0) = \alpha x_1(0) + \beta x_2(0)$ is satisfied.
- Step 2: it is easy to verify that

$$\dot{x} = \alpha \dot{x}_1 + \beta \dot{x}_2 = \alpha A x_1 + \beta A x_2 = A(\alpha x_1 + \beta x_2) = Ax.$$

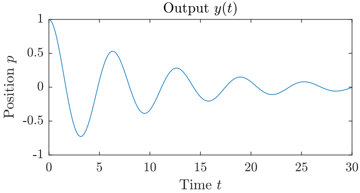
- Step 3:

$$y(t) = Cx(t) = C(\alpha x_1 + \beta x_2) = \alpha y_1(t) + \beta y_2(t)$$

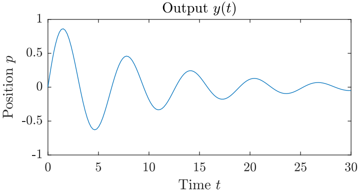
Case 2: Zero input $u(t) = 0$

Fact 2: Zero input $u(t) = 0$. The output $y(t)$ is linear in the initial conditions

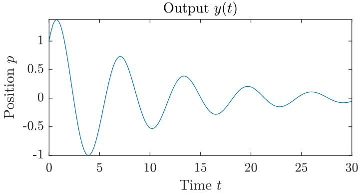
$$\begin{cases} x(0) \to y_1(t) \\ \hat{x}(0) \to y_2(t) \end{cases} \implies \alpha x(0) + \beta \hat{x}(0) \to \alpha y_1(t) + \beta y_2(t)$$



(a) Initial condition $x(0)$



(b) Initial condition $\hat{x}(0)$



(c) Initial condition $x(0) + \hat{x}(0)$

Why are LTI systems important?

- **Many important examples**
  - Electronic circuits (e.g., RLC circuits).
  - Many mechanical systems (e.g., spring-mass system).
- **Many important tools**
  - Step, impulse, and frequency responses. These are classical tools of control theory (developed in 1930's at Bell labs).
  - Classical control design toolbox, e.g., Bode/Nyquist plots, gain/phase margins, loop shaping.
  - Optimal control and estimators, e.g., linear quadratic regulator (LQR), and Kalman estimators.
  - Robust control design, e.g., $\mathcal{H}_2/\mathcal{H}_\infty$ control design and μ analysis for structural uncertainty.
- **Foundation to nonlinear system analysis and control** (linearization etc.)

Outline

Matrix exponential

Explore the initial condition response using the matrix exponential

- Consider a scalar system $\dot{x} = ax$ with initial condition $x(0) \in \mathbb{R}$.
- Its solution is $x(t) = e^{at}x(0)$.

Theorem

The solution to the homogeneous system of differential equations

$$\dot{x} = Ax, \qquad x(0) \in \mathbb{R}^n$$

is given by

$$x(t) = e^{At}x(0).$$

Definition

The matrix exponential is defined as

$$e^X = I + X + \frac{1}{2}X^2 + \frac{1}{3!}X^3 + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!}X^k. \tag{1}$$

where $X \in \mathbb{R}^{n\times n}$ and I is the $n \times n$ identity matrix.

Matrix exponential: Proof

- **Step 1**: Replacing X in (1) with At where $t \in \mathbb{R}$, we have

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k t^k.$$

- **Step 2**: differentiating this expression with respect to t gives

$$\begin{aligned}
\frac{d}{dt}e^{At} &= 0 + A + \frac{1}{1}A^2t + \frac{1}{2!}A^3t^2 + \ldots \\
&= A(I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \ldots) \\
&= A\sum_{k=0}^{\infty} \frac{1}{k!}A^k t^k \\
&= Ae^{At}.
\end{aligned}$$

- **Step 3**: we find that

$$\frac{d}{dt}\underbrace{e^{At}x(0)}_{x(t)} = A\underbrace{e^{At}x(0)}_{x(t)} \implies \frac{d}{dt}x(t) = Ax(t).$$

- **Final step**: initial condition $x(0) = e^{A\times 0}x(0)$ matches.

Corollary: linear in initial conditions

Theorem

The solution to the homogeneous system of differential equations

$$\dot{x} = Ax, \qquad x(0) \in \mathbb{R}^n$$

is give by

$$x(t) = e^{At}x(0).$$

It is immediate to see that the solution is linear in the initial condition

$$\begin{aligned} x(t) &= e^{At}(\alpha x_0 + \beta \hat{x_0}) \\ &= \alpha e^{At} x_0 + \beta e^{At} \hat{x}_0 \\ &= \alpha x_1(t) + \beta x_2(t) \end{aligned}$$

Example: double integrator

Example

Consider a simple second-order system

$$\ddot{q} = u, \qquad y = q.$$

- It is called a double integrator because $u(t)$ is integrated twice
- We write $x = (q, \dot{q})$, and

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

- By direct computation, we find $A^2 = 0$, and thus

$$e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

- When $u = 0$, the solution is

$$x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_1(0) + tx_2(0) \\ x_2(0) \end{bmatrix}$$

Example: Undamped oscillator

Example

Consider an spring-mass system with zero damping

$$\ddot{q} + \omega_0^2 q = u.$$

- Let $x_1 = q, x_2 = \dot{q}/\omega_0$. We have

$$\frac{d}{dt}x = \underbrace{\begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix}}_{A} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad \Longrightarrow \quad e^{At} = \begin{bmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\sin\omega_0 t & \cos\omega_0 t \end{bmatrix}$$

- This can be verified by differentiation

$$\frac{d}{dt}e^{At} = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix} \begin{bmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\sin\omega_0 t & \cos\omega_0 t \end{bmatrix} = Ae^{At}.$$

- The solution to the initial value problem is given by

$$x(t) = e^{At}x(0) = \begin{bmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\sin\omega_0 t & \cos\omega_0 t \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}.$$

Diagonalization

- Suppose the system
$$\dot{x} = Ax$$
has n distinct eigenvalues.
- Then, there exist an invertible matrix T such that TAT^{-1} is diagonal.
- **Coordinate change**: Choose a new coordinate $z = Tx$, then we have
$$\dot{z} = T\dot{x} = TAx = \underbrace{TAT^{-1}}_{\Lambda} z$$
- It is not difficult to show that
$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_n \end{bmatrix} \quad \Rightarrow \quad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & e^{\lambda_n t} \end{bmatrix}$$
- Implications for stability using eigenvalue test.
- Start from a real eigenvector $x(0) = v$; we have $x(t) = e^{At}v = e^{\lambda t}v$.

$\rightarrow$ a **mode** of the system.

Outline

Input/output responses

Consider a single-input and single output LTI system

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, and $y \in \mathbb{R}$.

- **Step input** (also known as Heaviside step function)

$$u(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t > 0 \end{cases}$$

- **Impulse input** (also known as delta function)

$$u(t) = p_\epsilon(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1/\epsilon & \text{if } 0 \leq t < \epsilon \\ 0 & \text{if } t \geq \epsilon \end{cases} \qquad \delta(t) = \lim_{\epsilon \to 0} p_\epsilon(t)$$

- **Frequency input** (also known as sinusoidal excitation)

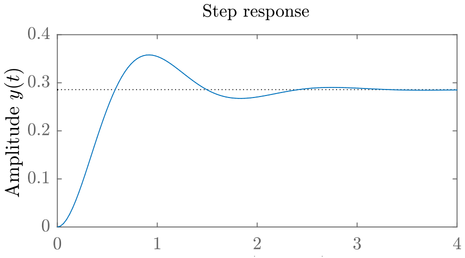
$$u(t) = \sin(\omega t + \phi).$$

Step response

Example (Open-loop stable system)

Consider an LIT system with

$$A = \begin{bmatrix} -1 & 4 \\ -3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0.$$

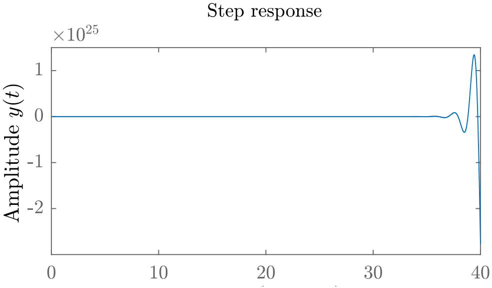


Step response

Example (Open-loop unstable system)

Consider an LIT system with

$$A_2 = \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0.$$



Step response

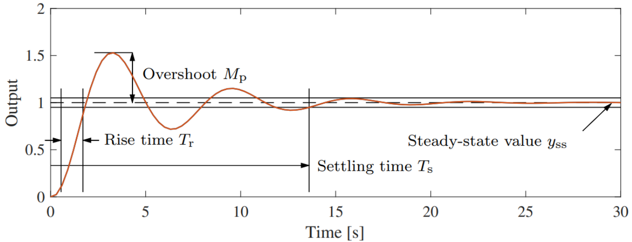


Figure: Sample step response. The rise time, overshoot, settling time, and steady-state value give the key performance properties of the signal.

- **Steady-state value** y_{ss}: final level of the output, assuming it converges
- **Rise time** T_r: time required for the signal to first go from 10% of its final value to 90% of its final value.
- **Overshoot** M_p: the percentage of the final value by which the signal initially rises above the final value
- **Settling time** T_s: time required for the signal to stay within 2% of its final value for all future times

Outline

Summary

- The output $y(t)$ of an LIT system has very nice **linear properties**:
  - Zero initial state $x(0) = 0$: *the output* $y(t)$ *is linear in input* $u(t)$;
  - Zero input $u(t) = 0$: *the output* $y(t)$ *is linear in initial states* $x(0)$.
- Initial response – **matrix exponential**:
  - The solution to $\dot{x} = Ax, x(0) \in \mathbb{R}^n$ is given by $x(t) = e^{At}x(0)$.
- Three very important test signals:
  - **Step input** $u(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t > 0 \end{cases}$
  - **Impulse input**

$$u(t) = p_\epsilon(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1/\epsilon & \text{if } 0 \leq t < \epsilon \\ 0 & \text{if } t \geq \epsilon \end{cases} \qquad \delta(t) = \lim_{\epsilon \to 0} p_\epsilon(t)$$

  - **Frequency input**

$$u(t) = \sin(\omega t + \phi).$$