ECE 171A: Linear Control System Theory
Lecture 11: Input/output responses (II)

Yang Zheng

Assistant Professor, ECE, UCSD

April 20, 2022

Reading materials: Ch 6.3, Ch 9.1

Outline

Input/output responses

Consider a single-input and single output LTI system

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, and $y \in \mathbb{R}$.

- **Step input** (also known as Heaviside step function)

$$u(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t > 0 \end{cases}$$

- **Impulse input** (also known as delta function)

$$u(t) = p_\epsilon(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1/\epsilon & \text{if } 0 \leq t < \epsilon \\ 0 & \text{if } t \geq \epsilon \end{cases} \qquad \delta(t) = \lim_{\epsilon \to 0} p_\epsilon(t)$$

- **Frequency input** (also known as sinusoidal excitation)

$$u(t) = \sin(\omega t + \phi).$$

Step response

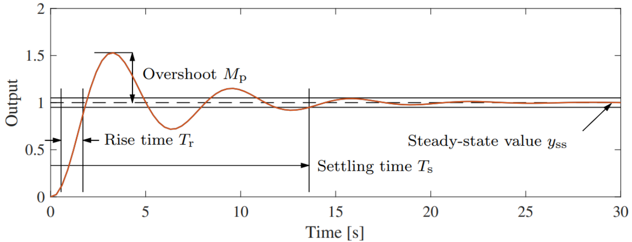


Figure: Sample step response. The rise time, overshoot, settling time, and steady-state value give the key performance properties of the signal.

- **Steady-state value** y_{ss}: final level of the output, assuming it converges
- **Rise time** T_r: time required for the signal to first go from 10% of its final value to 90% of its final value.
- **Overshoot** M_p: the percentage of the final value by which the signal initially rises above the final value
- **Settling time** T_s: time required for the signal to stay within 2% of its final value for all future times

Outline

Impulse response

Consider two LTI systems

- System 1: open-loop stable system

$$A_1 = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0.$$

- System 2: open-loop unstable system

$$A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0.$$

We consider the impulse response

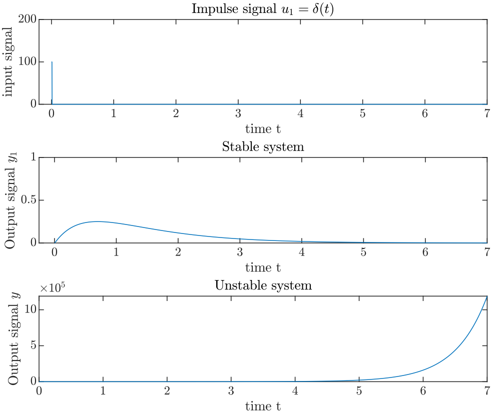
- **Impulse input** (also known as delta function)

$$u(t) = p_\epsilon(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1/\epsilon & \text{if } 0 \le t < \epsilon \\ 0 & \text{if } t \ge \epsilon \end{cases} \qquad \delta(t) = \lim_{\epsilon \to 0} p_\epsilon(t)$$

- In the simulation, we choose $\epsilon = 0.01$ seconds.

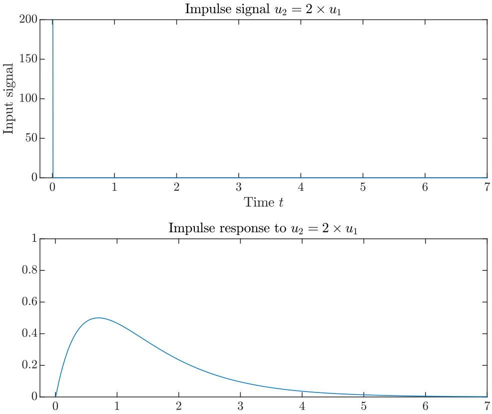
Impulse response - result 1

- Matlab command: `sys = ss(A, B, C, D); % create an LTI system`
- `y = lsim(sys,u,t,x0); % simulate the response to the input u`



Impulse response - result 2

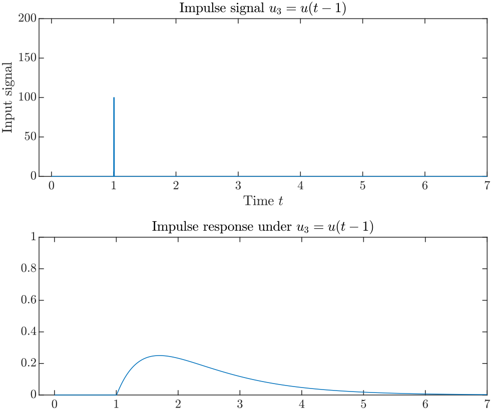
Case 2: Scale the input $u_2(t) = 2u_1(t) = 2\delta(t)$.



Q: Compared to the response in Case 1, what do you observe?

Impulse response - result 3

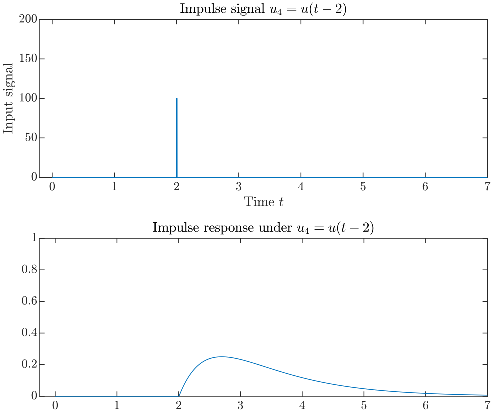
Case 3: Shift the input $u_3(t) = u_1(t-1) = \delta(t-1)$.



Q: Compared to the response in Case 1, what do you observe?

Impulse response - result 4

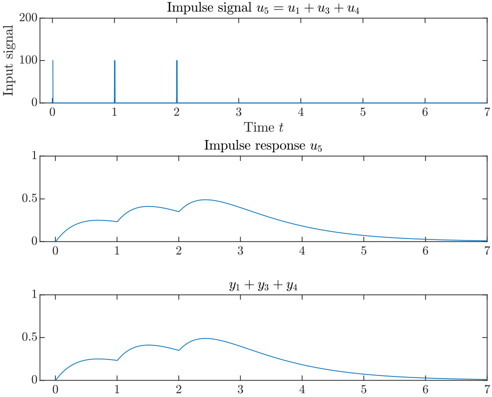
Case 4: Shift the input $u_4(t) = u_1(t-2) = \delta(t-2)$.



Q: Compared to the response in Case 1 and Case 3, what do you observe?

Impulse response - result 5

Case 5: Sum three inputs $u_5(t) = u_1(t) + u_3(t) + u_4(t)$.



Q: Compared to the response in Cases 1, 3, and 4, what do you observe?

Input/output response

Impulse response: the output of the system with zero initial condition and having an impulse $\delta(t)$ as its input can be written as

$$h(t) = \int_0^t Ce^{A(t-\tau)} B\delta(\tau) d\tau + D\delta(t) = Ce^{At}B + D\delta(t).$$

- **Informally**: any input $u(t)$ can be decomposed into a sum of impulses

$$u(t) = \sum_{\tau=0}^{\infty} u(\tau)\delta(t-\tau)$$

- The system response becomes

$$y(t) = \underbrace{Ce^{At}x(0)}_{\text{Initial response}} + \underbrace{\int_0^t h(t-\tau)u(\tau)d\tau}_{\text{forced response}}$$

(Forced response: superposition of the response to an infinite set of shifted impulses whose magnitudes are given by the input $u(t)$)

Outline

Frequency response

Frequency input (also known as sinusoidal excitation)

$$u(t) = \sin(\omega t + \phi).$$

Consider three LTI systems

- Open-loop stable system 1:

$$A_1 = \begin{bmatrix} -1 & 4 \\ -3 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_1 = 0.$$

- Open-loop stable system 2:

$$A_2 = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_2 = 0.$$

- Open-loop unstable system:

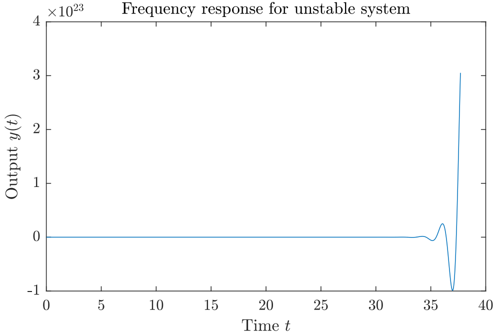
$$A_3 = \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_3 = 0.$$

Frequency response: unstable system

Frequency input (also known as sinusoidal excitation)

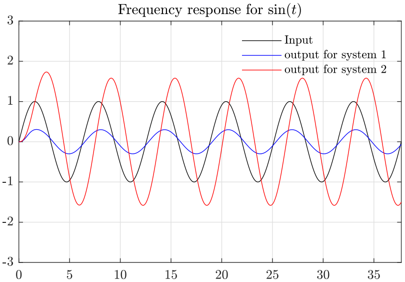
$$u(t) = \sin(\omega t + \phi).$$

Take $\omega = 1, \phi = 0$.

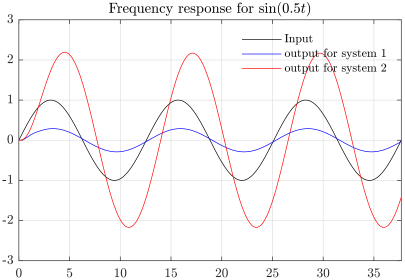


Frequency response: stable systems

Response to $u(t) = \sin(t)$



Response to $u(t) = \sin(0.5t)$

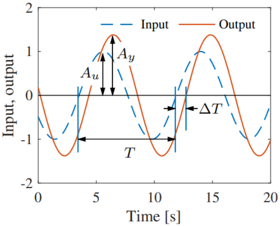


Frequency response - Bode plot

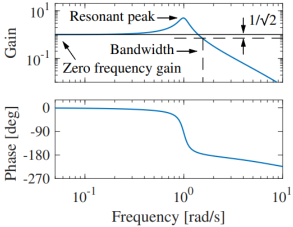
The steady output has a different **amplitude** plus a **shifted phase**.

- The **gain** is the ratio of the amplitudes of the sinusoids, which can be determined by measuring the height of the peaks.
- The **phase** is determined by comparing the ratio of the time between zero crossings of the input and output.

A convenient way to view the frequency response is to plot how the gain and phase depend on ω — Bode plot



(a) Input/output response



(b) Frequency response

Frequency response - Bode plot

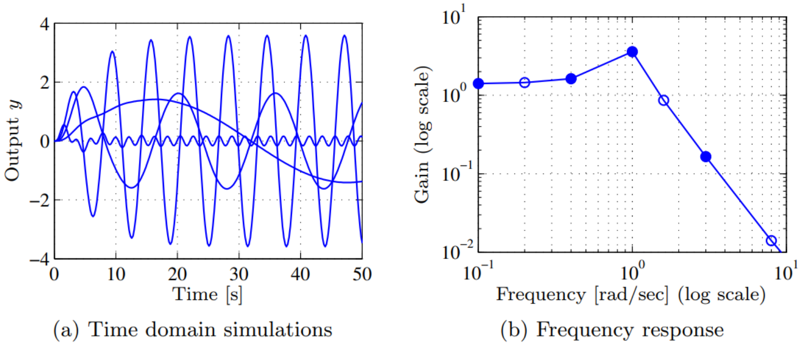


Figure: A frequency response (gain only) computed by measuring the response of individual sinusoids. The figure on the left shows the response of the system as a function of time to a number of different unit magnitude inputs (at different frequencies). The figure on the right shows this same data in a different way, with the magnitude of the response plotted as a function of the input frequency. The filled circles correspond to the particular frequencies shown in the time responses.

Outline

Solution to differential equation

Consider a state-space system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$.

Theorem

The solution to the linear differential equation (1) *is given by*

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau.$$

Proof by verification:

- **Step 1**: satisfy the initial condition?

$$x(0) = e^{A\times 0}x(0) + \int_0^0 e^{A(t-\tau)}Bu(\tau)d\tau = I \times x(0).$$

Proof

- **Step 2**: satisfy the differential question?

$$
\begin{aligned}
\frac{d}{dt}x(t) &= \frac{d}{dt}e^{At}x(0) + \frac{d}{dt}\left(\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right) \\
&= Ae^{At}x(0) + \frac{d}{dt}\left(\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right) \\
&= Ae^{At}x(0) + \frac{d}{dt}\left(e^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau\right) \\
\text{The Product Rule } \rightarrow &= Ae^{At}x(0) + Ae^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau + e^{At}\times e^{-At}Bu(t) \\
&= Ae^{At}x(0) + A\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Bu(t) \\
&= A\left(e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right) + Bu(t) \\
&= Ax(t) + Bu(t)
\end{aligned}
$$

- This verifies the convolution satisfies the differential equation (1).

The convolution equation

Theorem

The solution to the linear differential equation (1) *is given by*

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t). \tag{2}$$

- (2) is called the **convolution equation**.
- The output $y(t)$ is *jointly linear* in both the initial conditions $x(0)$ and the input $u(t)$, which follows from the linearity of matrix/vector multiplications and integration.
- All the linear properties of LTI systems in Lecture 10 can be directly proved from the general solution (2).
- The dynamics of the system, characterized by A, play an important role in both stability and performance.

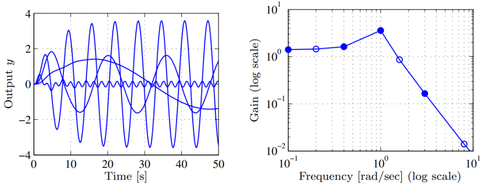
Outline

Summary

- **Impulse response**

$$h(t) = \int_0^t Ce^{A(t-\tau)}B\delta(\tau)d\tau + D\delta(t) = Ce^{At}B + D\delta(t).$$

- **Frequency responses**



(a) Time domain simulations (b) Frequency response

- **The convolution equation**

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t).$$

another version is $y(t) = \underbrace{Ce^{At}x(0)}_{\text{Initial response}} + \underbrace{\int_0^t h(t-\tau)u(\tau)d\tau}_{\text{forced response}}$