ECE 171A: Linear Control System Theory
Lecture 12: Transfer functions (I)

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Reading materials: Ch 6.3, Ch 9.1, 9.2

Announcements

- **Office hours**
  - Recall from Lecture 1: "Ideally, I would like most of you, if not all, go to the office hours together even if you don't have questions. You can even help us answering questions to others. It is important to have a community for this class!"
- **Piazza**: Check it regularly, and feel free to ask questions (Lectures, textbook, HW etc.)
- **Midterm I**: Many of you did well (Send us an email if you want to chat)
  - Maximum: 102
  - Above 90: 10; 80 - 90: 10
  - Mean: 67.3; Median: 69
- Anonymous survey on workload and feedback; Please spend 2 minutes filling it out by Thursday night.

`https://forms.gle/pfBzhbjDpC3A9rWr5`

Recall from Lecture 1: You will need to invest a significant amount of time, so that you will enjoy this course and learn a lot.

Outline

The convolution equation

Consider a state-space system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$.

Theorem

The solution to the linear differential equation (1) *is given by*

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t). \tag{2}$$

Some observations:

- Control design is to design an input signal $u(t)$ to shape $y(t)$ (stability, tracking performance, less overshot, less oscillation, robustness, etc.);
- The solution (2) is too complex to use for designing a controller;
- We look for some elegant and simple tools: make the mapping from $u(t)$ to $y(t)$ easier to compute — **Transfer functions**

Steady-state response

A common practice in evaluating the response of a linear system is to separate out the **short-term response** from the **long-term response**.

- **Transient response:** which occurs in the first period of time after the input is applied.

  It reflects the mismatch between the initial condition and the steady-state solution

- **Steady-state response:** which is the portion of the output response that reflects the long-term behavior of the system under the given inputs.
  - For inputs that are periodic, the steady-state response will often be periodic (e.g., **frequency response**)
  - For constant inputs, the response will often be constant (e.g., **step response**)

- Mathematical derivation will be discussed in Lecture 12.

Example

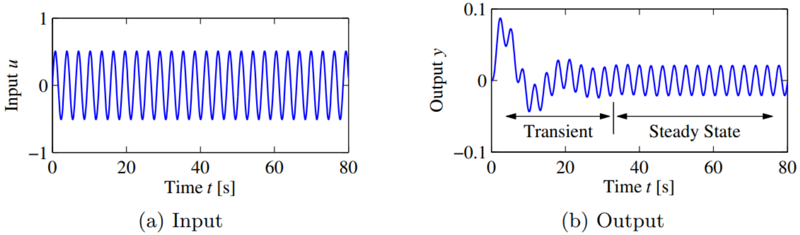


Figure 6.8: Transient versus steady-state response. The input to a linear system is shown in (a), and the corresponding output with $x(0) = 0$ is shown in (b). The output signal initially undergoes a transient before settling into its steady-state behavior.

Step response

Step input (also known as Heaviside step function)

$$u(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t > 0 \end{cases}$$

- Let's assume $x(0) = 0$
- Solve the step response

$$\begin{aligned} y(t) &= Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \\ &= \int_0^t Ce^{A(t-\tau)}Bd\tau + D = -\int_t^0 Ce^{A\rho}Bd\rho + D \\ &= C\int_0^t e^{A\rho}d\rho B + D = C\left(A^{-1}e^{A\rho}\Big|_{\rho=0}^{\rho=t}\right)B + D \\ &= \underbrace{CA^{-1}e^{At}B}_{\text{transient}} + \underbrace{D - CA^{-1}B}_{\text{steady-state}} \end{aligned}$$

Outline

Frequency response

The **frequency response** of an input/output system measures the way in which the system responds to a sinusoidal excitation.

- The particular solution associated with a sinusoidal excitation is itself a sinusoid at the same frequency.
- We can compare the **magnitude** and **phase** of the output sinusoid to the input (— **Transfer function** and **Bode plot**).

Let us consider a sinusoid input

$$u(t) = \cos \omega t.$$

- Evaluating (2) with input $u(t) = \cos \omega t$ can be very messy.
- We use the fact that the system is linear to simplify the derivation.
- In particular, Euler's formula tells us that

$$\cos \omega t = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

- Thanks to the linearity, we can use the exponential input $u(t) = e^{st}$, and then construct the solution by letting $s = i\omega$ and $s = -i\omega$.

Frequency response - derivation

- We apply the convolution equation to $u = e^{st}$

$$\begin{aligned} y(t) &= Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)} B e^{s\tau} d\tau + De^{st} \\ &= Ce^{At}x(0) + Ce^{At}\int_0^t e^{(sI-A)\tau} d\tau B + De^{st} \end{aligned}$$

- We assume $(sI - A)$ is invertible, then

$$\begin{aligned} y(t) &= Ce^{At}x(0) + Ce^{At}\left((sI-A)^{-1} e^{(sI-A)\tau}\Big|_{\tau=0}^{\tau=t} \right) B + De^{st} \\ &= Ce^{At}x(0) + Ce^{At}(sI-A)^{-1}\left(e^{(sI-A)t} - I \right) B + De^{st} \\ &= Ce^{At}x(0) + C(sI-A)^{-1}e^{st}B - Ce^{At}(sI-A)^{-1}B + De^{st} \end{aligned}$$

- Finally, we obtain

$$y(t) = \underbrace{Ce^{At}\left(x(0) - (sI-A)^{-1}B\right)}_{\text{transient}} + \underbrace{\left(C(sI-A)^{-1}B + D\right)e^{st}}_{\text{steady-state}}$$

Frequency response - steady-state component

The solution corresponding to the exponential input e^{st} is

$$y(t) = \underbrace{Ce^{At}\left(x(0) - (sI - A)^{-1}B\right)}_{\text{transient}} + \underbrace{\left(C(sI - A)^{-1}B + D\right)e^{st}}_{\text{steady-state}}$$

- If the system matrix A is stable, the transient component decays to zero
- **The steady-state component** is proportional to the exponential input e^{st}.
- We can write the steady-state response as

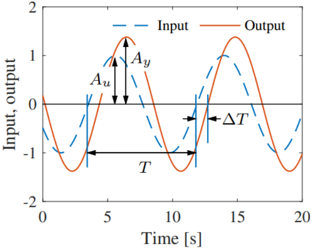
$$y_{\text{ss}}(t) = Me^{i\theta}e^{st} = Me^{st+i\theta},$$

  where $G(s) = C(sI - A)^{-1}B + D \quad \leftarrow$ **Transfer function**
- When $s = i\omega$, we call
  - $M = |G(i\omega)|$ the **gain**, and
  - $\theta = \arg(G(i\omega))$ the **phase** of the system at the forcing frequency ω

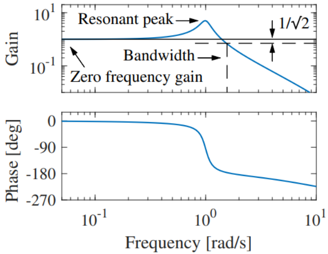
Frequency response - steady-state component

The steady-state solution for a sinusoid $u = \cos \omega t$ is given by

$$y_{\text{ss}}(t) = \text{Re}\left(G(i\omega)e^{i\omega t}\right) = |G(i\omega)| \cos(\omega t + \angle G(i\omega))$$



(a) Input/output response



(b) Frequency response

Figure 6.11: Steady-state response of an asymptotically stable linear system to a sinusoid. (a) A sinusoidal input of magnitude A_u (dashed) gives a sinusoidal output of magnitude A_y (solid), delayed by ΔT seconds. (b) Frequency response, showing gain and phase. The gain is given by the ratio of the output amplitude to the input amplitude, $M = A_y/A_u$. The phase lag is given by $\theta = -2\pi\Delta T/T$; it is negative for the case shown because the output lags the input.

Some terminology

- **Zero frequency gain**: The gain of a system at ω, corresponds to the ratio between a constant input and the steady output

$$M_0 = G(0) = -CA^{-1}B + D.$$

  - The zero frequency gain is well defined only if A is invertible.
  - Zero frequency gain is a relevant quantity only for stable systems.
  - In EE, the zero frequency gain is often called the **DC gain**.

- The **bandwidth** ω_b of a system is the frequency range over which the gain has decreased by no more than a factor $1/\sqrt{2}$ from its reference value (either *zero-frequency gain* or *high-frequency gain*)

- **Resonant peak** M_{r}, the largest value of the frequency response

- **Peak frequency** ω_{mr}, the frequency where the maximum occurs
  - The frequency of the sinusoidal input that produce the largest possible output and the gain at the frequency.

Outline

Transfer functions

Transfer functions: transmission of exponential Signals e^{st} with $s = \sigma + i\omega$

$$e^{st} = e^{\sigma t} e^{i\omega t} = e^{\sigma t}(\cos \omega t + i \sin \omega t)$$

where $\sigma \leq 0$: decay rate.

- Find the transfer function for the state-space system

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du. \tag{3}$$

- The output $y(t)$ of system (3) to the input $e^{\sigma t}$ is

$$y(t) = \underbrace{Ce^{At}\left(x(0) - (sI - A)^{-1}B\right)}_{\text{transient}} + \underbrace{\left(C(sI - A)^{-1}B + D\right)e^{st}}_{\text{steady-state}}$$

- The **transfer function** from u to y of the system (3) is the coefficient of the term e^{st}, i,e,,

$$G(s) = C(sI - A)^{-1}B + D.$$

Example: calculating transfer function

Example

Consider an LTI system

$$\dot{x}_1 = -a_1 x_1 - a_2 x_2 + u \qquad y = x_2$$
$$\dot{x}_2 = x_1$$

- The system matrices are

$$A = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}, D = 0.$$

- Compute its transfer function

$$\begin{aligned} G(s) = C(sI - A)^{-1}B + D &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s + a_1 & a_2 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{s^2 + a_1 s + a_2} \begin{bmatrix} s & -a_2 \\ 1 & s + a_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{s^2 + a_1 s + a_2}. \end{aligned}$$

Example: computing steady-state responses

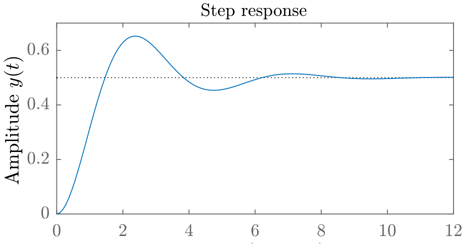
Example

- Suppose $a_1 = 1, a_2 = 2$. Its transfer function is

$$G(s) = \frac{1}{s^2 + s + 2}.$$

- The steady-state response to a step input $u(t) = 1$ is e^{st} with $s = 0$, i.e.

$$y_{\text{ss}} = G(0)u = \frac{1}{2}.$$



Example: computing steady-state responses

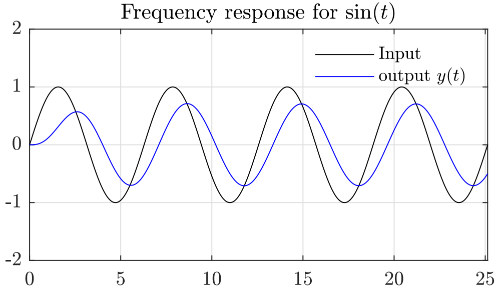
Example

- The steady-state response to a sin input $u(t) = \sin \omega t$ is

$$y = M \sin(\omega t + \phi), \quad \text{where} M = |G(i\omega)|, \theta = \arg(G(i\omega))$$

- **Case 1**: $u(t) = \sin t \ \rightarrow \ y(t)$?

$$G(i\omega) = \frac{1}{(i\omega)^2 + i\omega + 2}, \qquad M = |G(i)| = \frac{1}{\sqrt{2}}, \quad \theta = -45^\circ$$



Example: computing steady-state responses

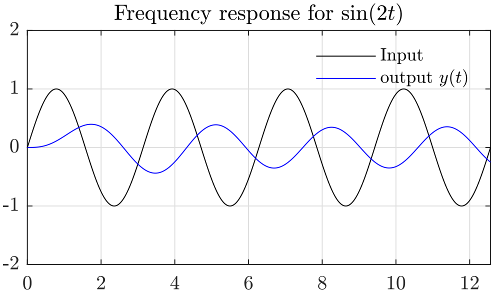
Example

- The steady-state response to a sin input $u(t) = \sin \omega t$ is

$$y = M \sin(\omega t + \phi), \quad \text{where} M = |G(i\omega)|, \theta = \arg(G(i\omega))$$

- **Case 2**: $u(t) = \sin 2t \;\to\; y(t)$?

$$G(i2) = \frac{1}{-2 + i2}, \qquad M = |G(i2)| = \frac{1}{2\sqrt{2}}, \quad \theta = -135^\circ$$



Outline

Frequency response

Determining transfer functions

Summary

Summary

- Transient response and steady-state response

$$y(t) = \underbrace{Ce^{At}\left(x(0) - (sI - A)^{-1}B\right)}_{\text{transient}} + \underbrace{\left(C(sI - A)^{-1}B + D\right)e^{st}}_{\text{steady-state}}$$

- Transfer function

$$G(s) = C(sI - A)^{-1}B + D.$$

  - Express the steady-state solution of a stable linear system forced by a sinusoidal input

$$u(t) = \sin(\omega t) \to y_{\text{ss}} = |G(i\omega)| \sin(\omega t + \angle G(i\omega))$$

- **Frequency domain modeling**: Modeling a system through its response to sinusoidal and exponential signals.
  - We represent the dynamics of the system in terms of the generalized frequency s rather than the time domain variable t.
  - The **transfer function** provides a complete representation of a linear system in the frequency domain.