ECE 171A: Linear Control System Theory
Lecture 13: Transfer functions (II)

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Reading materials: Ch 9.2, 9.3, 9.4

Outline

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Convolution equation and Transfer functions

Consider a state-space system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$.

- The solution to the state-space system (1) is given by

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t). \tag{2}$$

Transfer functions:

- We apply the convolution equation to $u = e^{st}$

$$y(t) = \underbrace{Ce^{At}\left(x(0) - (sI - A)^{-1}B\right)}_{\text{transient}} + \underbrace{\left(C(sI - A)^{-1}B + D\right)e^{st}}_{\text{steady-state}}$$

- The transfer function for the state-space system (1) is

$$G(s) = C(sI - A)^{-1}B + D$$

Transfer functions - overview

- **Transfer functions** — A compact description of the input/output relation for a linear time-invariant (LTI) system.
- Combining transfer functions with **block diagrams** gives a powerful algebraic method to analyze linear systems with many blocks.
- The transfer function allows new interpretations of system dynamics.
- **Many graphical tools**, such as the Bode plot (a powerful graphical representation of the transfer function that was introduced by Bode.)

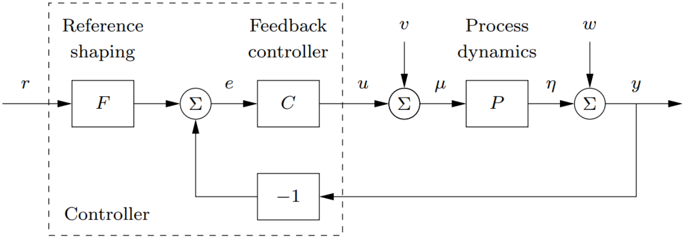


Figure: A block diagram for a feedback control system

Response to periodic inputs

The basic idea of the transfer function comes from looking at the frequency response of a system.

$$G(s) = C(sI - A)^{-1}B + D \quad \leftarrow \quad \textbf{Transfer function}$$

- Suppose that we have an input signal that is periodic. We can then decompose it

$$u(t) = \sum_{k=0}^{\infty} \left(a_k \sin(k\omega_\mathrm{f} t) + b_k \cos(k\omega_f t)\right)$$

- The output will be sine and cosine waves, with possibly **shifted magnitude and phase**, which can be determined by

$$G(i\omega) = C(i\omega - A)^{-1}B + D,$$

  where $\omega = k\omega_\mathrm{f}, k = 1, \ldots, \infty$.

- Thanks to linearity (**superposition**), the final steady-state response will be a sum of these signals.

The exponential input e^{st}

- The transfer function generalizes this notion to allow a broader class of input signals besides periodic ones.
- The transfer function can also be introduced as the ratio of the *Laplace transforms* of the output and the input.

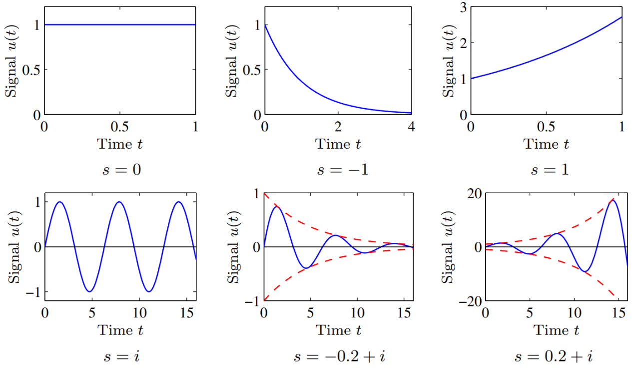


Figure: Examples of exponential signals. The top row: exponential signals with a real exponent, and the bottom row: those with complex exponents.

Frequency-domain modeling

Frequency domain modeling: Modeling a system through its response to sinusoidal and exponential signals.

- We represent the dynamics of the system in terms of the generalized frequency s rather than the time domain variable t.
- The transfer function provides a complete representation of a linear system in the frequency domain.

Some benefits of transfer functions:

- Provide a particularly **convenient representation** in manipulating and analyzing complex linear feedback systems.
- **Graphical representations** (Bode and Nyquist plots) that capture interesting properties of the underlying dynamics — Weeks 5/6
- We can introduce concepts that express the degree of stability of a system – **stability margins**, Week 6
- Express the changes/uncertainty in a system because of modeling error, considering sensitivity to process variations – **robustness**, Week 9

Outline

Linear ODEs

Consider a linear system described by the controlled differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \ldots + b_0 u, \quad (3)$$

where u is the input, and y is the output.

- We aim to determine the transfer function of (3) (i.e., input/output relationship in frequency domain);
- Let the input $u(t) = e^{st}$, and since the system is linear, the output is $y(t) = y_0 e^{st}$.
- Plug $u(t) = e^{st}$ and $y(t) = y_0 e^{st}$ into (3),

$$(s^n + a_{n-1}s^{n-1} + \ldots + a_0)y_0 e^{st} = (b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0)e^{st}$$

- We now have

$$y(t) = y_0 e^{st} = \underbrace{\frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}}_{G(s)} e^{st}.$$

Example: Cruise control

Example

The system dynamics are given by

$$\dot{p} = v(t), \qquad \dot{v}(t) = \frac{1}{m}u(t).$$

where p denotes the position, v denotes the velocity of the vehicle.

- It is the same as
$$\ddot{p} = \frac{1}{m}u(t).$$
- Applying an exponential input $u = e^{st}$ leads to
$$s^2 p_0 e^{st} = \frac{1}{m}e^{st} \qquad \Rightarrow \qquad s^2 y(t) = \frac{1}{m}u(t).$$
- The input/output relationship between $p(t)$ and $u(t)$ (i.e., transfer function) in the frequency domain is
$$G(s) = \frac{1}{ms^2}.$$

Example: spring-mass system

Example

The system dynamics are given by

$$\ddot{x}(t) + 2\zeta\omega_0\dot{x}(t) + \omega_0^2 x(t) = u(t),$$

where $x(t)$ denotes the position of the mass, ζ is the damping coefficient, and ω_0 denotes the natural frequency.

- Applying an exponential input $u = e^{st}$ leads to

$$\begin{aligned} & s^2 x_0 e^{st} + 2\zeta\omega_0 s x_0 e^{st} + \omega_0^2 x_0 e^{st} = e^{st} \\ \Rightarrow \quad & (s^2 + 2\zeta\omega_0 s + \omega_0^2) x(t) = u(t). \end{aligned}$$

- The input/output relationship between $x(t)$ and $u(t)$ (i.e., transfer function) in the frequency domain is

$$G(s) = \frac{1}{s^2 + 2\zeta\omega_0 s + \omega_0^2}.$$

Example: Vibration damper

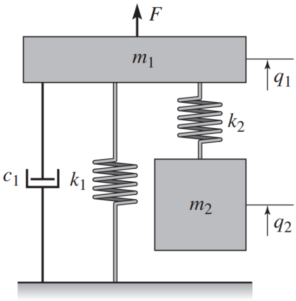


Figure: A vibration damper. Vibrations of the mass m_1 can be damped by providing it with an auxiliary mass m_2, attached to m_1 by a spring with stiffness k_2. The parameters m_2 and k_2 are chosen so that the frequency $\sqrt{k_2/m_2}$ matches the frequency of the vibration.

Example: Vibration damper

Example

The system dynamics are given by

$$\begin{aligned} m_1\ddot{q}_1 + c_1\dot{q}_1 + k_1 q_1 + k_2(q_1 - q_2) &= F, \\ m_2\ddot{q}_2 + k_2(q_2 - q_1) &= 0. \end{aligned}$$

- **Objective**: determine the transfer function from the force F to the position q_1.
- We first find particular exponential solutions

$$\begin{aligned} (m_1 s^2 + c_1 s + k_1)q_1 + k_2(q_1 - q_2) &= F \\ m_2 s^2 q_2 + k_2(q_2 - q_1) &= 0. \end{aligned}$$

- Eliminate q_2 and we have the transfer function

$$G_{q_1 F}(s) = \frac{m_2 s^2 + k_2}{m_1 m_2 s^4 + m_2 c_1 s^3 + (m_1 k_2 + m_2(k_1 + k_2))s^2 + k_2 c_1 s + k_1 k_2}$$

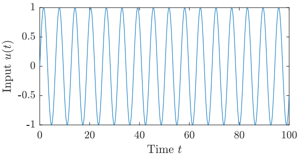
- The transfer function has a **zero** at $s = \pm i\sqrt{k_2/m_2}$ — **Blocking property**

Blocking property

Parameters $m_1 = 1, c_1 = 1, k_1 = 1, m_2 = 1, k_2 = 1.$

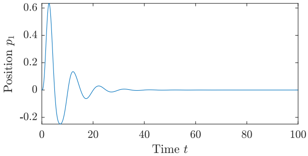
- **Case 1: external input**

$$u = \sin(\omega t), \qquad \text{with } \omega = 1.$$

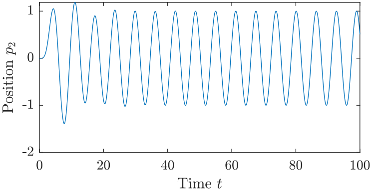


(a) Input $u = \sin(t)$

$\Rightarrow$



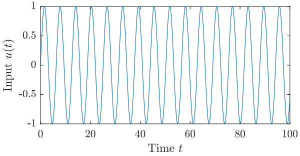
(b) Position of mass 1



(c) Postion of mass 2

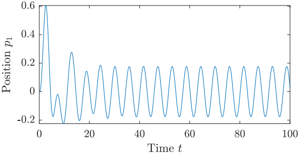
Other frequencey responses

- **Case 2: external input** $u = \sin(\omega t)$, with $\omega = 1.1$.



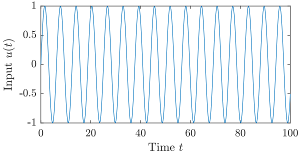
(a) Input $u = \sin(1.1t)$

$\Rightarrow$



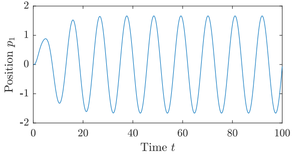
(b) Position of mass 1

- **Case 3: external input** $u = \sin(\omega t)$, with $\omega = 0.578$.



(a) Input $u = \sin(1.1t)$

$\Rightarrow$



(b) Position of mass 1

Common transfer functions

Type	System	Transfer function
Integrator	$\dot{y} = u$	$\frac{1}{s}$
Differentiator	$y = \dot{u}$	s
First-order system	$\dot{y} + ay = u$	$\frac{1}{s+a}$
Double integrator	$\ddot{y} = u$	$\frac{1}{s^2}$
Damped oscillator	$\ddot{y} + 2\zeta\omega_0\dot{y} + \omega_0^2 y = u$	$\frac{1}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$
State-space system	$\dot{x} = Ax + Bu$ $y = Cx + Du$	$C(sI - A)^{-1}B + D$
PID controller	$y = k_\mathrm{p}u + k_\mathrm{d}\dot{u} + k_\mathrm{i}\int u$	$k_\mathrm{p} + k_\mathrm{d}s + \frac{k_\mathrm{i}}{s}$
Time delay	$y(t) = u(t-\tau)$	$e^{-\tau s}$

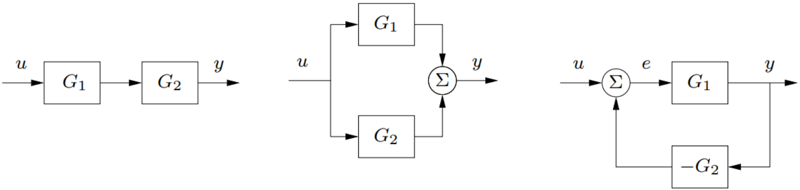
Table: Transfer functions for some common linear time-invariant systems.

Outline

Block diagrams

The combination of block diagrams and transfer functions is a powerful way to represent control systems.

- Input-output relationship can be derived by **algebraic manipulations** of the transfer functions.



(a) $G_{yu} = G_2 G_1$ (b) $G_{yu} = G_1 + G_2$ (c) $G_{yu} = \dfrac{G_1}{1 + G_1 G_2}$

Figure: Interconnections of linear systems. **Series** (a), **parallel** (b), and **feedback** (c) connections are shown.

Feedback connection

- It is easy to see the relationship

$$y = G_1 e, \qquad e = u - G_2 y$$

- Elimination of e gives

$$\begin{aligned} y = G_1(u - G_2 y) \quad &\Rightarrow \quad (1 + G_1 G_2) y = G_1 u \\ &\Rightarrow \quad y = \frac{G_1}{1 + G_1 G_2} u \end{aligned}$$

- The transfer function of the feedback connection is thus

$$G = \frac{G_1}{1 + G_1 G_2}.$$

These three basic interconnections can be used as the basis for computing transfer functions for more complicated systems.

Outline

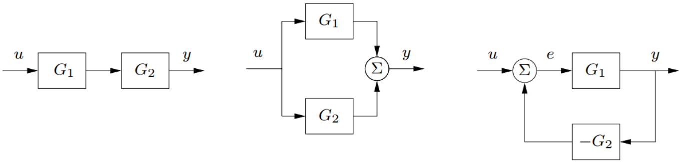
Summary

- **Frequency domain modeling**: Modeling a system through its response to sinusoidal and exponential signals.
  - We represent the dynamics of the system in terms of the generalized frequency s rather than the time domain variable t.
  - The **transfer function** provides a complete representation of a linear system in the frequency domain.
- **Transfer function for linear ODEs**

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \ldots + b_0 u,$$

$$G(s) = \frac{b_m s^m + b_{m-1}s^{n-1} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}.$$

- **Block diagram with transfer functions**



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