ECE 171A: Linear Control System Theory
Lecture 14: Zeros, Poles and Bode plot

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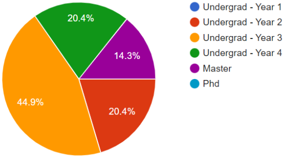
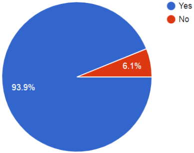
April 29, 2022

Reading materials: Ch 9.5, 9.6

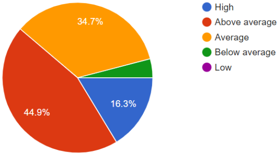
Survey Feedback



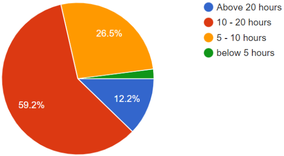
(a) Years



(b) First Control course



(c) Work load



(d) Time each week

- 10 - 15 hours per week are reasonable
- Homework and office hours are very important!
- You should think about HW questions and read lecture notes by yourself first, and then come to office hours prepared (you can work on HW here).

Survey Feedback

Q: Which aspect(s) of this course have you particularly enjoyed or valued so far? Any other comments on the course [1]

- "Feeling super smart after understanding the material. Attending office hours and collaborating."
- "I have enjoyed the office hours, it makes the class feel like a community"
- "I love the way this course is run, going to office hours is something I actually look forward to. I've been able to make some friends in the class while in office hours which is actually the first time that's happened since my time here at UCSD."
- "Overall, I think this course surprised me with how engaging and fun control theory can be"
- "The MATLAB explanations have been better than I've ever had in any of my past courses."
- "I have enjoyed the lectures and the matlab examples, even though they give me the most trouble. I also like the math we do in this class."
- "Learning that aspects of controls are abundant in our lives. However translating them into mathematical models and functional code is challenging."
- . . . . . .

[1] These are copied from the answers. If you do not want them to be here, I'll remove them.

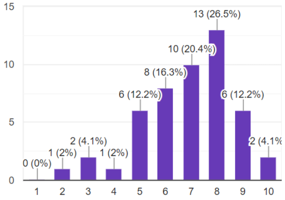
Survey Feedback

Q: Which aspect(s) of this course do you think could be improved or changed for the rest of this quarter?[2]

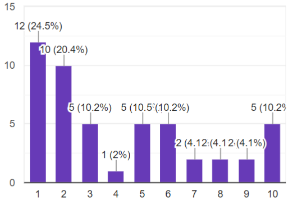
- The pace of this class is a little fast. Speak slower
- Sometimes lectures run long causing the next lecture to start halfway through the previous lecture and compounding.
- Too much homework
- I don't really find proving the theory to be that interesting. I think that understanding the content to apply it is sufficient.
- Can you upload annotated lecture after the lectures.
- Please don't put a homework the same week of the midterm
- Office hours do not appear to be too structured; When a lot of people come, there is not a lot of room and it is very cramped.
- Having some office hours online
- Some mechanism where the final would replace some aspect of the midterm if we scored higher, since it would promote growth of students over the course more effectively.
- ......

[2] These are adapted from the answers. If you do not want them to be here, I'll remove them.

Feedback on Midterm I



(a) Difficulty



(b) Satisfactory

Q: Comments on Midterm I

- More time or less problems; ran out of time; no time to check answers etc.
- This might be the best written test I've seen at UCSD so far. It was challenging and I did bad, but I know that everything I got wrong on the exam I should have known.

Q: Expectations on Midterm II

- finish all the problems and do them correctly.
- I would like to select 10 in "How satisfied are you regarding the outcome of Midterm II?"

Grading

- Recall from Lecture 1:
  - "A standard grade scale (e.g., 93%+ = A) will be used with a curve based on the class performance (e.g., if the top students have grades in the 83%-86% range, then this will correspond to letter grade A)"
- You are not competing with your classmates/friends.
- Grades are important; what you have really learned is also very important.
- Take a maximum of the following methods
  1. HW 40% + Midterm I 10 % + Midterm II 10 % + Final 40 %
  2. HW 40% + Midterm I 5 % + Midterm II 10 % + Final 45 %
  3. HW 40% + Midterm I 10 % + Midterm II 5 % + Final 45 %
- So that "promote growth of students over the course more effectively.", "It seems like a lot of this class build on top of each other and if that is the case. I feel like we shouldn't be faulted too much by the end of the class if we are really able to grasp the material well.";

Outline

Outline

Laplace transforms

- The traditional way to find transfer functions is to use Laplace transforms.
- Only a few elementary properties of Laplace transforms are needed for basic control applications;

Definition

Given a time-domain function $f(t) : \mathbb{R}^+ \to \mathbb{R}$, the Laplace transform maps f to a function $F = \mathcal{L}(f) : \mathbb{C} \to \mathbb{C}$ of a complex variable

$$F(s) = \int_0^\infty e^{-st} f(t) dt, \qquad s \in \mathbb{C}.$$

- **Linearity**:

$$\begin{aligned}\mathcal{L}(af + bg) &= \int_0^\infty e^{-st} (af(t) + bg(t)) dt \\ &= a \int_0^\infty e^{-st} f(t) dt + b \int_0^\infty e^{-st} g(t) dt \\ &= a\mathcal{L}(f) + b\mathcal{L}(g).\end{aligned}$$

Laplace transforms

- **Differentiation**:
$$\mathcal{L}\left(\frac{dx(t)}{dt}\right) = s\mathcal{L}(x(t)) - x(0).$$
Proof: integration by parts:
$$\int_0^\infty \frac{d}{dt}\left(x(t)e^{-st}\right)dt = x(t)e^{-st}\Big|_0^\infty = -x(0)$$
$$= \int_0^\infty \frac{d}{dt}\left(x(t)\right)e^{-st}dt + \int_0^\infty x(t)\frac{d}{dt}\left(e^{-st}\right)dt$$
$$= \mathcal{L}\left(\frac{dx(t)}{dt}\right) - s\mathcal{L}(x(t))$$
- **Integration**
$$\mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \frac{1}{s}\mathcal{L}\left(f(t)\right).$$
- **Laplace transform of selected functions**
$$\mathcal{L}\left(e^{-at}\right) = \int_0^\infty e^{-(s+a)t}dt = \frac{-1}{s+a}e^{-(s+a)t}\Big|_0^\infty = \frac{1}{s+a}.$$

Laplace transforms of selected functions

- delta function $\delta(t)$: we have $\mathcal{L}(\delta(t)) = 1$.
- Step input

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \quad \Rightarrow \quad \mathcal{L}(u(t)) = \frac{1}{s}.$$

Convolution in time domain gives multiplication in frequency domain!

- Suppose $y(t) = \int_0^\infty h(t-\tau)u(\tau)d\tau$.
- Then

$$\begin{aligned} Y(s) := \mathcal{L}(y(t)) &= \int_0^\infty \int_0^\infty h(t-\tau)u(\tau)e^{-st}d\tau dt \\ &= \int_0^\infty \int_0^\infty h(\mu)u(\tau)e^{-s\tau}e^{-s\mu}d\tau d\mu \\ &= \int_0^\infty h(\mu)e^{-s\mu}d\mu \times \int_0^\infty u(\tau)e^{-s\tau}d\tau \\ &= H(s)U(s) \end{aligned}$$

Application

- Consider a state-space system

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du$$

- Zero initial condition $x(0) = 0$.
- Take Laplace transform for both sides, and we have

$$\begin{aligned} sX(s) &= AX(s) + BU(s), \\ Y(s) &= CX(s) + DU(s) \end{aligned} \quad \Rightarrow \quad Y(s) = \underbrace{\left(C(sI - A)^{-1}B + D\right)}_{G(s)} U(s)$$

The transfer function is

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D.$$

This is the same as **the steady-state response to exponential input signal**.

Outline

Transfer functions

Type	System	Transfer function
Integrator	$\dot{y} = u$	$\frac{1}{s}$
Differentiator	$y = \dot{u}$	s
First-order system	$\dot{y} + ay = u$	$\frac{1}{s+a}$
Double integrator	$\ddot{y} = u$	$\frac{1}{s^2}$
Damped oscillator	$\ddot{y} + 2\zeta\omega_0\dot{y} + \omega_0^2 y = u$	$\frac{1}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$
State-space system	$\dot{x} = Ax + Bu$ $y = Cx + Du$	$C(sI - A)^{-1}B + D$

The **features** of a transfer function are often associated with **important system properties**.

- zero frequency gain
- the locations of the poles and zeros.

Zero frequency gain

Zero frequency gain: the magnitude of the transfer function at $s = 0$.

- **Interpretation**: The ratio of the steady-state value of the output with respect to a step input (which can be represented as $u = e^{st}$ with $s = 0$).

Examples:

- State-space model:

$$G(s) = C(sI - A)^{-1}B + D \qquad \Rightarrow \qquad G(0) = -CA^{-1}B + D$$

- Linear differential equation:

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}y}{dt^{m-1}} + \ldots + b_0 u.$$

$$\Rightarrow \qquad G(0) = \frac{b_0}{a_0}.$$

- Integrator $\dot{y} = u$, $G(s) = \frac{1}{s}$: we have $G(0) = \infty$. $\rightarrow$ **pole**
- Differentiator $y = \dot{u}$, $G(s) = s$: we have $G(0) = 0$. $\rightarrow$ **zero**

Poles and zeros

Consider a linear system with the rational transfer function

$$G(s) = \frac{b(s)}{a(s)}.$$

- **Poles**: The roots of the polynomial $a(s)$.
- **Zeros**: The roots of the polynomial $b(s)$.

Interpretation of poles:

- Consider a linear differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}y}{dt^{m-1}} + \ldots + b_0 u. \quad (1)$$

- Let $u = 0$ (no external force; homogeneous equation). If p is a pole, i.e., p is a solution to

$$s^n + a_{n-1}s^{n-1} + \ldots + a_0 = 0$$

- then, $y(t) = e^{pt}$ is a solution to (1).
- **A pole p corresponds to a mode of the system, i.e., $y(t) = e^{pt}$.**

Poles and zeros

Interpretation of zeros:

- Consider an exponential input e^{st}
- The exponential output is $y(t) = G(s)e^{st}$.
- If $G(s) = 0$, then the (steady-state) output is zero.

Zeros of transfer function thus block transmission of the corresponding exponential signals.

Example (Vibration dampers)

$$G_{q_1 F}(s) = \frac{m_2 s^2 + k_2}{m_1 m_2 s^4 + m_2 c_1 s^3 + (m_1 k_2 + m_2(k_1 + k_2))s^2 + k_2 c_1 s + k_1 k_2}$$

- The transfer function has a **zero** at $s = \pm i\sqrt{k_2/m_2}$ — **Blocking property**

Example: Vibration damper

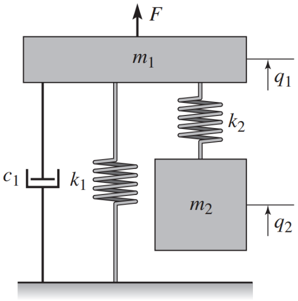


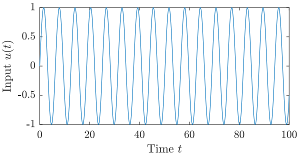
Figure: A vibration damper. Vibrations of the mass m_1 can be damped by providing it with an auxiliary mass m_2, attached to m_1 by a spring with stiffness k_2. The parameters m_2 and k_2 are chosen so that the frequency $\sqrt{k_2/m_2}$ matches the frequency of the vibration.

Blocking property

Parameters $m_1 = 1, c_1 = 1, k_1 = 1, m_2 = 1, k_2 = 1.$

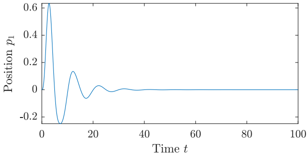
- **The following external input is blocked**

$$u = \sin(\omega t), \qquad \text{with } \omega = 1.$$

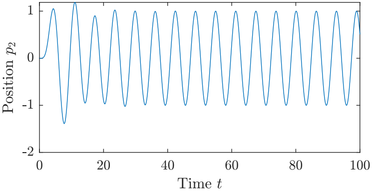


(a) Input $u = \sin(t)$

$\Rightarrow$



(b) Position of mass 1



(c) Postion of mass 2

Some connections

State-space models vs. **transfer function representations**
(assuming SISO system)

	State-space model	Transfer function
Model	$\dot{x} = Ax + Bu$ $y = Cx + Du$	$G(s) = C(sI - A)^{-1}B + D = \frac{b(s)}{a(s)}$
Variables	input $u(t) \in \mathbb{R}$, output $y(t) \in \mathbb{R}$, state $x(t) \in \mathbb{R}^n$	input $u(t) \in \mathbb{R}$, output $y(t) \in \mathbb{R}$,
Stability	Poles (eigenvalues) of A	Poles of $G(s)$

Poles (eigenvalues) of the matrix A = Poles of the transfer function $G(s)$

- The inverse of $(sI - A)$ can be computed by

$$(sI - A)^{-1} = \frac{1}{\det(sI - A)} \mathrm{adj}(sI - A).$$

Some connections

- adj denotes the adjugate of a matrix [3]

$$(\mathrm{adj}(sI - A))_{ij} = (-1)^{i+j} M_{ji}$$

  where M_{ji} denote the determinant of the $(n-1) \times (n-1)$ matrix that results from deleting row j and column i of A.

- Then, we have

$$G(s) = C(sI - A)^{-1}B + D = C\frac{\mathrm{adj}(sI - A)}{\det(sI - A)}B + D.$$

Pole excess (also known as **relative degree**): the difference between the number of poles and zeros $n_{\mathrm{pe}} = n - m$.

- A rational transfer function is called *proper* if $n_{\mathrm{pe}} \geq 0$;
- A rational transfer function is called *strictly proper* if $n_{\mathrm{pe}} > 0$;
- If $n < m$, there is no state space realization (thus it is *improper*).

[3] See https://en.wikipedia.org/wiki/Adjugate_matrix for 2×2 and 3×3 examples.

Pole zero diagram

Pole-zero diagram: A convenient way to view the poles and zeros of a transfer function.

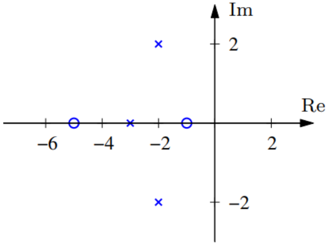


Figure: A pole zero diagram for a transfer function with zeros at -5 and -1 and poles at -3 and $-2 \pm 2j$. The circles represent the locations of the zeros, and the crosses the locations of the poles.

- **Stable poles**: Poles in the left half-plane
- **Unstable poles**: Poles in the right half-plane

Outline

Bode plot

The **frequency response** of a linear system can be computed from its transfer function by setting $s = i\omega$, i.e.,

$$u(t) = e^{i\omega t} = \cos(\omega t) + i \sin(\omega t).$$

- The resulting output is

$$y(t) = G(i\omega)e^{i\omega t} = Me^{i(\omega t + \theta)} = M\cos(\omega t + \theta) + iM\sin(\omega t + \theta)$$

- Thus, we have $\cos(\omega t) \to M\cos(\omega t + \theta)$ and $\sin(\omega t) \to M\sin(\omega t + \theta)$

The frequency response $G(i\omega)$ can be represented by two curves — **Bode plot**

- **Gain curve**: gives $|G(i\omega)|$ as a function of frequency ω — log/log scale (traditionally often in dB — $20\log|G(i\omega)|$; but we use $\log|G(i\omega)|$)
- **Phase curve**: gives $\angle G(i\omega)$ as a function of frequency ω — log/linear scale in degrees

Sketching Bode plots

- Part of the popularity of Bode plots is that they are easy to sketch and interpret.
- Since the frequency scale is logarithmic, they cover the behavior of a linear system over a wide frequency range.

Consider a transfer function

$$G(s) = \frac{b_1(s)b_2(s)}{a_1(s)a_2(s)}$$

- **Gain curve**: simply adding and subtracting gains corresponding to terms in the numerator and denominator

$$\log|G(s)| = \log|b_1(s)| + \log|b_2(s)| - \log|a_1(s)| - \log|a_2(s)|.$$

- **Phase curve**: similarly we have

$$\angle G(s) = \angle b_1(s) + \angle b_2(s) - \angle a_1(s) - \angle a_2(s).$$

Bode plot - Blocks

A polynomial can be written as a product of terms of the type

$$k, \qquad s, \qquad s+a, \qquad s^2 + 2\zeta\omega_0 s + \omega_0^2$$

- Sketch Bode diagrams for these terms;
- Complex systems: add the gains and phases of the individual terms

Case 1: $G(s) = s^k$ — Two special cases: $k = 1$, a differentiator; $k = -1$, an integrator

$$\log |G(s)| = k \times \log \omega, \qquad \angle G(i\omega) = k \times 90^\circ$$

- The gain curve is a straight line with slope k, and the phase curve is a constant at $k \times 90^\circ$
- The case when $k = 1$ corresponds to a differentiator and has slope 1 with phase 90°
- The case when $k = -1$ corresponds to an integrator and has slope -1 with phase -90°

Case 1: $G(s) = s^k$

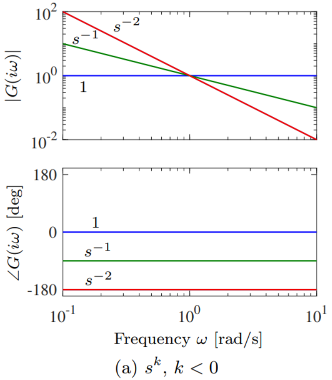
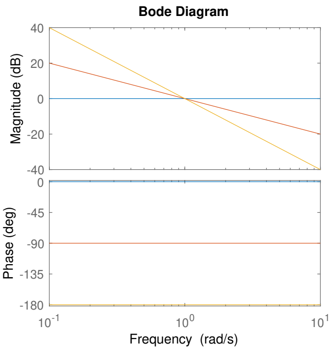
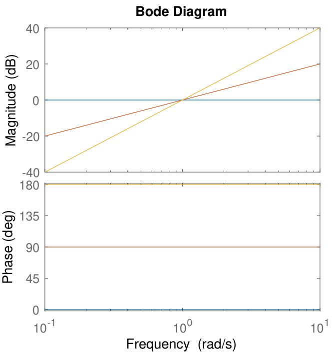


Figure: Bode plots of the transfer functions $G(s) = s^k$ for $k = -2, -1, 0, 1, 2$. On a log-log scale, the gain curve is a straight line with slope k. The phase curves for the transfer functions are constants, with phase equal to $k \times 90^\circ$.

Case 1: $G(s) = s^k$



(a) $s^k, k < 0$



(b) $s^k, k > 0$

Figure: Bode plots of the transfer functions $G(s) = s^k$ for $k = -2, -1, 0, 1, 2$ — from Matlab

```
G0 = tf([1],[1]); % create a transfer function
G1 = tf([1 0],[1]); % create a transfer function
W = {0.1,10}; bode(G0,G1,W); % Bode plot
```

Outline

Summary

- The **features** of a transfer function are often associated with **important system properties**.
  - zero frequency gain
  - the locations of the poles and zeros: Poles — modes of a system; Zeros – Block transmission of certain signals

Poles (eigenvalues) of the matrix A = Poles of the transfer function $G(s)$

- The frequency response $G(i\omega)$ can be represented by two curves — **Bode plot**
  - **Gain curve**: gives $|G(i\omega)|$ as a function of frequency ω — log/log scale (often in dB — $20 \log |G(i\omega)|$)
  - **Phase curve**: gives $\angle G(i\omega)$ as a function of frequency ω — log/linear scale in degrees