ECE 171A: Linear Control System Theory
Lecture 15: Bode plot and Routh-Hurwitz stability

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Reading materials: Ch 9.6, Ch 2.2

Outline

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Bode plot

The **frequency response** of a **stable** linear system can be computed from its transfer function by setting $s = i\omega$, i.e.,

$$u(t) = e^{i\omega t} = \cos(\omega t) + i \sin(\omega t).$$

- The resulting steady-state output is

$$y(t) = G(i\omega)e^{i\omega t} = Me^{i(\omega t+\theta)} = M\cos(\omega t + \theta) + iM\sin(\omega t + \theta)$$

- Thus, we have $\cos(\omega t) \to M\cos(\omega t + \theta)$ and $\sin(\omega t) \to M\sin(\omega t + \theta)$

The frequency response $G(i\omega)$ can be represented by two curves — **Bode plot**

- **Gain curve**: gives $|G(i\omega)|$ as a function of frequency ω — log/log scale (traditionally often in dB — $20\log|G(i\omega)|$; but we mainly use $\log|G(i\omega)|$)
- **Phase curve**: gives $\angle G(i\omega)$ as a function of frequency ω — log/linear scale in degrees

Sketching Bode[1] plots

Hendrik Wade Bode (1905 - 1982): a pioneer of modern control theory and electronic telecommunications.

- Part of the popularity of Bode plots is that they are easy to sketch and interpret.
- Since the frequency scale is logarithmic, they cover the system behavior over a wide frequency range.

Consider a transfer function

$$G(s) = \frac{b_1(s)b_2(s)}{a_1(s)a_2(s)}$$

- **Gain curve**: simply adding and subtracting gains corresponding to terms in the numerator and denominator

$$\log|G(s)| = \log|b_1(s)| + \log|b_2(s)| - \log|a_1(s)| - \log|a_2(s)|.$$

- **Phase curve**: similarly we have

$$\angle G(s) = \angle b_1(s) + \angle b_2(s) - \angle a_1(s) - \angle a_2(s).$$

[1] https://en.wikipedia.org/wiki/Hendrik_Wade_Bode

Bode plot — Blocks

A polynomial can be written as a product of terms of the type

$$k, \qquad s, \qquad s+a, \qquad s^2 + 2\zeta\omega_0 s + \omega_0^2$$

- Sketch Bode diagrams for these terms;
- Complex systems: add the gains and phases of the individual terms

Case 1: $G(s) = s^k$ — Two special cases: $k = 1$, a differentiator; $k = -1$, an integrator

$$\log |G(s)| = k \times \log \omega, \qquad \angle G(i\omega) = k \times 90^\circ$$

- The gain curve is a straight line with slope k, and the phase curve is a constant at $k \times 90^\circ$
- The case when $k = 1$ corresponds to a differentiator and has slope 1 with phase 90°
- The case when $k = -1$ corresponds to an integrator and has slope -1 with phase -90°

Case 1: $G(s) = s^k$

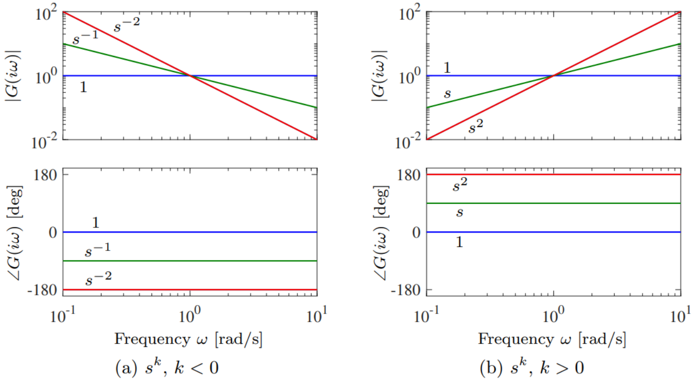
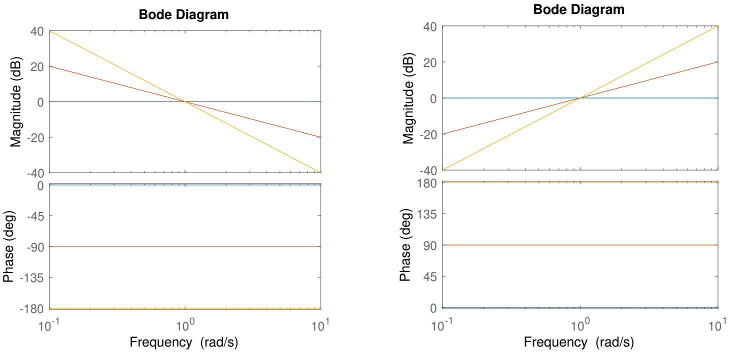


Figure: Bode plots of the transfer functions $G(s) = s^k$ for $k = -2, -1, 0, 1, 2$. On a log-log scale, the gain curve is a straight line with slope k. The phase curves for the transfer functions are constants, with phase equal to $k \times 90°$.

Case 1: $G(s) = s^k$

```
G0 = tf([1],[1]); % create a transfer function
G1 = tf([1 0],[1]); % create a transfer function
W = {0.1,10}; bode(G0,G1,W); % Bode plot
```



(a) $s^k, k < 0$ (b) $s^k, k > 0$

Figure: Bode plots of the transfer functions $G(s) = s^k$ for $k = -2, -1, 0, 1, 2$ — from Matlab

Case 2: first-order system

Consider the transfer function of a first-order system

$$G(s) = \frac{a}{s+a}, \qquad a > 0.$$

- We have

$$|G(s)| = \frac{|a|}{|s+a|}, \qquad \angle G(s) = \angle a - \angle(s+a).$$

- The gain curve is

$$|G(i\omega)| = \log a - \frac{1}{2}\log(\omega^2 + a^2) \approx \begin{cases} 0, & \text{if } \omega < a \\ \log a - \log \omega, & \text{if } \omega > a \end{cases}$$

- The phase curve is

$$\angle G(i\omega) = -\frac{180}{\pi}\arctan\frac{\omega}{a} \approx \begin{cases} 0, & \text{if } \omega < \frac{a}{10} \\ -45 - 45(\log\omega - \log a), & \text{if } a/10 < \omega < 10a \\ -90, & \text{if } \omega > 10a \end{cases}$$

Case 2: first-order system

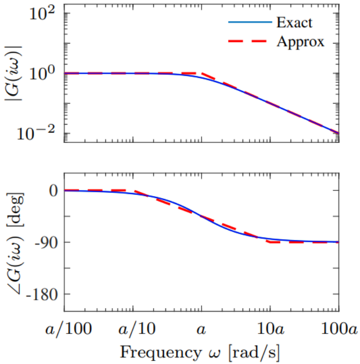


Figure: Bode plot of the first-order system $G(s) = a/(s + a)$, which can be approximated by asymptotic curves (dashed) in both the gain and the frequency, with the breakpoint in the gain curve at $\omega = a$ and the phase decreasing by 90° over a factor of 100 in frequency.

A first-order system behaves like a constant for low frequencies and like an integrator for high frequencies.

Case 3: second-order system

Consider the transfer function of a first-order system

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}, \qquad 0 < \zeta < 1.$$

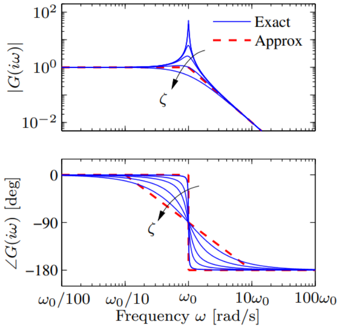
- The gain curve is

$$|G(i\omega)| = 2\log\omega_0 - \frac{1}{2}\log(\omega^4 + 2\omega_0^2\omega^2(2\zeta^2 - 1) + \omega_0^4)$$
$$\approx \begin{cases} 0, & \text{if } \omega \ll \omega_0 \\ 2\log\omega_0 - 2\log\omega, & \text{if } \omega \gg \omega_0 \end{cases}$$

- The largest gain $Q = \max_\omega |G(i\omega)| \approx 1/(2\zeta)$, called the Q-value, is obtained for $\omega \approx \omega_0$ – **Resonant frequency**
- The phase curve is

$$\angle G(i\omega) = -\frac{180}{\pi}\arctan\frac{2\zeta\omega_0\omega}{\omega_0^2 - \omega} \approx \begin{cases} 0, & \text{if } \omega \ll \omega_0 \\ -180, & \text{if } \omega \gg \omega_0 \end{cases}$$

Case 3: Second-order system



(b) Second-order system

Figure: Bode plot of the second-order system $G(s) = \omega_0^2/(s^2 + 2\zeta\omega_0 s + \omega_0^2)$, which has a peak at frequency ω_0 and then a slope of -2 beyond the peak; the phase decreases from 0° to -180°. The height of the peak and the rate of change of phase depending on the damping ratio ζ ($\zeta = 0.02, 0.1, 0.2, 0.5$, and 1.0 shown).

The asymptotic approximation is poor near $\omega = \omega_0$ and that the Bode plot depends strongly on ζ near this frequency.

Poles and zeros in the right Half Plane

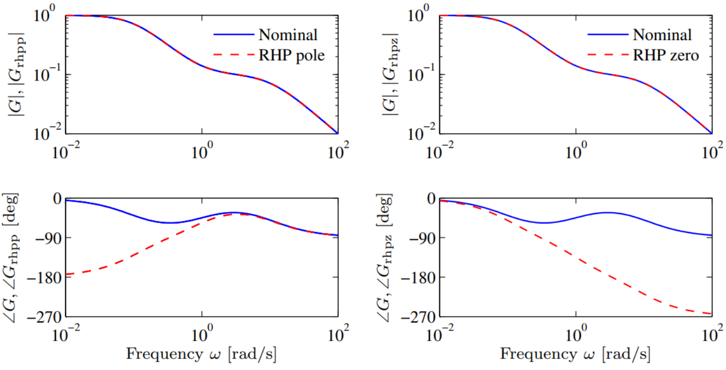
Example

Consider the transfer functions

$$\begin{aligned} G(s) &= \frac{s+1}{(s+0.1)(s+10)}, \\ G_{\text{rhpp}}(s) &= \frac{s+1}{(s-0.1)(s+10)}, \\ G_{\text{rhpz}}(s) &= \frac{-s+1}{(s+0.1)(s+10)}. \end{aligned}$$

- The gain curve of a transfer function remains the same if a pole or a zero is shifted from the left half-plane to the right half-plane.
- The phase will, however, change significantly as is illustrated by the example above.

Example



(a) Right half-plane pole (b) Right half-plane zero

- **Time delay** $G(s) = e^{-s\tau}$ represents a even more striking example of a change in phase than a right half-plane zero.
- Extra phase will cause difficulties for control since there is a delay between applying an input and seeing its effect — **fundamental limits** in Week 10

Outline

System insights

The Bode plot gives a quick overview of a **stable** linear system. Since many useful signals can be decomposed into a sum of sinusoids, it is possible to *visualize* the behavior of a system for different frequency ranges.

$$u(t) = \sin(\omega t) \qquad \rightarrow \qquad y_{\text{ss}} = |G(i\omega)| \sin(\omega t + \angle G(i\omega))$$

- The system can be viewed as a **filter**: change the input signals according to frequency range
- **Type 1: Lower-pass filter**, for example

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

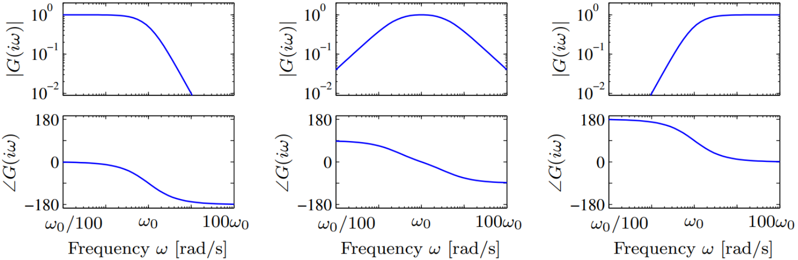
- **Type 2: Band-pass filter**, for example

$$G(s) = \frac{2\zeta\omega_0 s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

- **Type 3: High-pass filter**, for example

$$G(s) = \frac{s^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

Filters



$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(a) Low-pass filter

$$G(s) = \frac{2\zeta\omega_0 s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(b) Band-pass filter

$$G(s) = \frac{s^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(c) High-pass filter

Figure: Bode plots for low-pass, band-pass, and high-pass filters. The upper plots are the gain curves and the lower plots are the phase curves. Each system passes frequencies in a different range and attenuates frequencies outside of that range.

Example: Spring-mass system

Example

Consider a spring-mass with input u (force) and output q (position) as follows

$$m\ddot{q} + c\dot{q} + kq = u \qquad \rightarrow \qquad G(s) = \frac{1}{ms^2 + cs + k}$$

- **Case 1**: When s is small, we have

$$G(s) \approx \frac{1}{k} \qquad \rightarrow \qquad q = \frac{u}{k}$$

  which implies that for low-frequency inputs, the system behaves like a **spring** driven by a force.

- **Case 2**: When s is large, we have

$$G(s) \approx \frac{1}{ms^2} \qquad \rightarrow \qquad \ddot{q} = \frac{u}{m}$$

  which implies that the system behaves like a **mass** driven by a force (double integrator).

Example: Spring-mass system

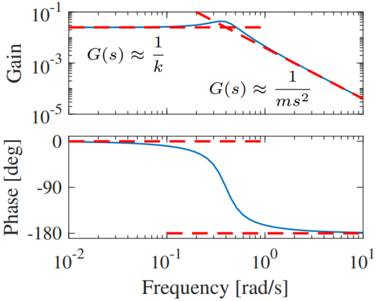
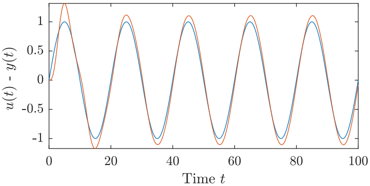


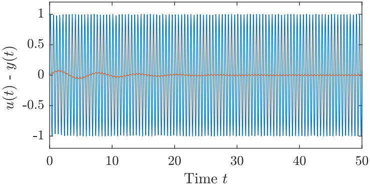
Figure: Bode plot for a spring–mass system. At low frequency the system behaves like a spring with $G(s) \approx 1/k$ and at high frequency the system behaves like a pure mass with $G(s) \approx 1/(ms^2)$

Example: Spring-mass system

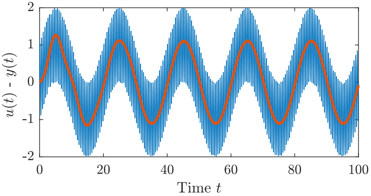
Consider parameters $m = 1; k = 1; c = 0.2;$



(a) Low frequency



(b) High frequency

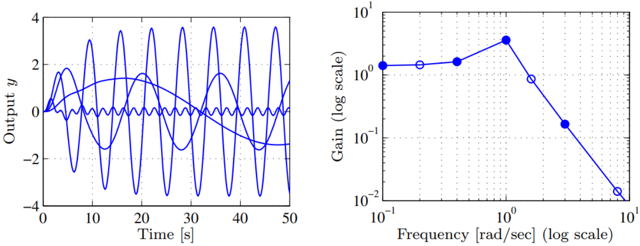


(c) Mixed frequencies

Determine Transfer function experimentally

Model a given application by measuring the frequency response

- Apply a sinusoidal signal at a fixed frequency.
- Measure the amplitude ratio and phase lag when steady state is reached.
- The complete frequency response is obtained by sweeping over a range of frequencies.



(a) Time domain simulations (b) Frequency response

Figure: A frequency response (gain only) computed by measuring the response of individual sinusoids.

Outline

Stability

Theorem (Stability of a linear system (Lyapunov sense))

The system $\dot{x} = Ax$ *is*

- **asymptotically stable** *if and only if all eigenvalues of* A *have a strictly negative real part, i.e.,* $\mathrm{Re}(\lambda_i) < 0$
- **unstable** *if any eigenvalues* A *has a strictly positive real part.*

Consider an LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du \end{aligned} \qquad \Longleftrightarrow \qquad G(s) = C(sI - A)^{-1}B + D$$

Poles (eigenvalues) of the matrix A **= Poles of the transfer function** $G(s)$

- A system is **bounded-input bounded-output (BIBO)** stable if every bounded input $u(t)$ leads to a bounded output $y(t)$.
- **BIBO stable**: if all poles of $G(s)$ are in the open left half-plane in the s domain (i.e., having negative real parts).

Routh-Hurwitz Criterion

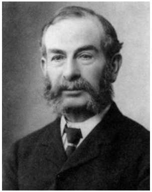
- **Eigenvalues** or **poles**

$$G(s) = \frac{b(s)}{a(s)}, \qquad a(s) = \det(sI - A)$$

- In the 1870s-1890s, **Edward Routh** (English Mathematician, 1831 – 1907) and **Adolf Hurwitz** (German Mathematician, 1859 – 1919) independently
  - developed a method for determining the **locations** in the s plane but **not the actual values** of the roots of a polynomial with constant real coefficients
- Characteristic polynomial:

$$a(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0$$

- **The Routh-Hurwitz method**
  - constructs a table with $n+1$ rows from the coefficients a_i of a polynomial $a(s)$
  - relates **the number of sign changes** in the first column of the table to **the number of roots** in the closed right half-plane



E. Routh

A. Hurwitz

Routh Table

- $a(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0$

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$	a_0
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$	0
s^{n-2}	$b_{n-1} = -\frac{\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}}$	$b_{n-3} = -\frac{\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{a_{n-1}}$	b_{n-5}	$\cdots$	0
s^{n-3}	$c_{n-1} = -\frac{\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}}{b_{n-1}}$	$c_{n-3} = -\frac{\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_{n-1} & b_{n-5} \end{vmatrix}}{b_{n-1}}$	c_{n-5}	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
s^0	a_0	0	0	$\cdots$	0

- Any row can be multiplied by a positive constant without changing the result

Routh-Hurwitz BIBO Stability Criterion

Theorem

Consider a Routh table from the polynomial $a(s)$ in

$$G(s) = \frac{b(s)}{a(s)}.$$

- *The number of sign changes in the first column of the Routh table is equal to the number of roots of $a(s)$ in the closed right half-plane.*

Corollary (BIBO Stability of LTI Systems)

The system $G(s)$ is **BIBO stable** *if and only if there are no sign changes in the first column of its Routh table.*

There are two special cases related to the Routh table:

1. The first element of a row is 0 but some of the other elements are not
   - **Solution**: replace the 0 with an arbitrary small ϵ
2. All elements of a row are 0 (not required in this course)

Example: Second-order System

Example

Consider the characteristic polynomial of a second-order system:

$$a(s) = as^2 + bs + c$$

- The Routh table is:

s^2	a	c
s^1	b	0
s^0	$-\frac{1}{b}(0 - bc) = c$	0

- A **necessary and sufficient condition** for BIBO stability of a second-order system is that all coefficients of the characteristic polynomial are non-zero and have the same sign.

Example: Third-order System

Example

Consider the characteristic polynomial of a third-order system:

$$a(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

- The Routh table is:

s^3	a_3	a_1
s^2	a_2	a_0
s^1	$-\frac{1}{a_2}(a_3 a_0 - a_1 a_2)$	0
s^0	a_0	0

- If $a_3 > 0$, then a **sufficient and necessary condition** for BIBO stability (all eigenvalues have strictly negative real parts) is

$$a_3 > 0, \qquad a_2 > 0, \qquad a_1 a_2 > a_0 a_3, \qquad a_0 > 0$$

- If $a_1 a_2 = a_0 a_3$, one pair of roots lies on the imaginary axis in the s plane and the system is marginally stable. This results in an all zero row in the Routh table.