ECE 171A: Linear Control System Theory

Lecture 16: Loop transfer functions and Nyquist plot

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Reading materials: Ch 10.1, 10.2

Announcement

- **Midterm exam II**: May 18 (Wednesday, Week 8)

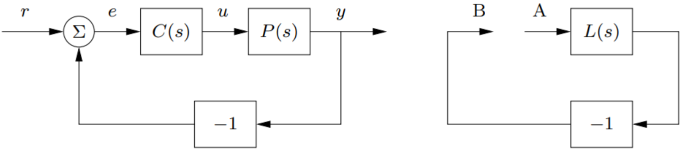
	Apr 22	Midterm I - in class		Homework 4
5	Apr 25	L12: Transfer function (I)	Ch 9.1, 9.2	
	Apr 25	D5: Review on complex numbers		
	Apr 27	L13: Transfer function (II)	Ch 9.2, 9.3, 9.4	
	Apr 29	L14: Poles, zeros and Bode plot	Ch 9.5, Ch 9.6	Homework 5
6	May 02	L15: Bode plot and Routh-Hurwitz stability	Ch 2.2, Ch 9.6	
	May 02	D6: Bode plot examples		
	May 04	L16: Loop transfer functions and Nyquist plot	Ch 10.1, 10.2	
	May 06	L17: Nyquist criterion and Stability margins	Ch 10.3, Ch 10.4	Homework 6
7	May 09	L18: Bode's relations and Root locus	Ch 10.4, Ch 12.5	
	May 09	D7: Nyquist plot examples		
	May 11	L19: PID control (I)	Ch 11.1, 11.2	
	May 13	L20: PID control (II)	Ch 11.3, 11.4	Homework 7
8	May 16	L21: Review		
	May 16	D8: Review, HWs, Q&A		
	May 18	Midterm II - in class		

- **Office hours**: week 6 - week 10
  - Tuesdays 4:30 pm - 6:20 pm (Yang Zheng, Jacobs Hall Room 4506)
  - Tuesdays 6:30 pm - 7:30 pm (Yang Zheng, Virtual, zoom link: `https://ucsd.zoom.us/j/95103569286`)
  - Thursdays 7:00 pm - 9:00 pm (Dehao Dai, Jacobs Hall Room 4506)

Outline

Outline

Stability of feedback systems



(a) Closed loop system (b) Open loop system

- **Lyapunov stability** — eigenvalue test of the closed-loop matrix; e.g.,

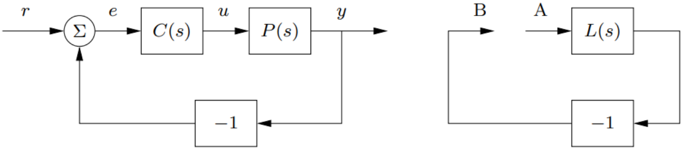
$$\begin{aligned} \text{Dynamics} \rightarrow \quad & \dot{x} = Ax + Bu, \\ \text{Feedback controller} \rightarrow \quad & u = -Kx \end{aligned} \quad \Rightarrow \quad \dot{x} = (A - BK)x.$$

- **Poles or The Routh–Hurwitz Criterion**;

$$\begin{cases} P(s) & = \dfrac{n_{\mathrm{p}}(s)}{d_{\mathrm{p}}(s)} \\ C(s) & = \dfrac{n_{\mathrm{c}}(s)}{d_{\mathrm{c}}(s)} \end{cases} \quad \Rightarrow \quad G_{yr}(s) = \frac{PC}{1+PC} = \frac{n_{\mathrm{p}}(s)n_{\mathrm{c}}(s)}{d_{\mathrm{p}}(s)d_{\mathrm{c}}(s) + n_{\mathrm{p}}(s)n_{\mathrm{c}}(s)}$$

They are **straightforward but give little guidance** for design: it is not easy to tell how the controller should be modified to make an unstable system stable.

Loop analysis



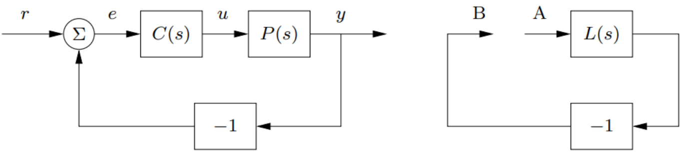
(a) Closed loop system (b) Open loop system

Figure: **The loop transfer function** $L(s) = P(s)C(s)$. The stability of the feedback system (a) can be determined by tracing signals around the loop.

- We break the loop in (b) and ask whether a signal injected at the point A has the same magnitude and phase when it reaches point B.
- Determine **stability and robustness** of closed loop systems by investigating how sinusoidal signals propagate around the feedback loop.
- Reason about the **closed loop behavior** of a system through the frequency domain properties of the **open loop transfer function**.

The second very important graphical tool — the **Nyquist stability theorem**.

Nyquist's idea



(a) Closed loop system (b) Open loop system

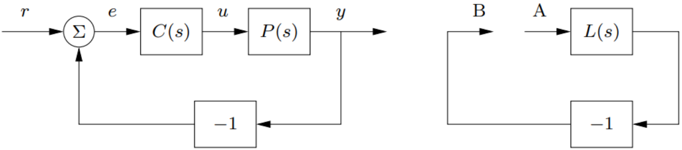
- Nyquist's idea was to first investigate conditions under which oscillations can occur in a feedback loop.
- The **Loop transfer function**:

$$L(s) = P(s)C(s).$$

- Assume that a sinusoid of frequency ω_0 is injected at point A. In steady state, the signal at point B will also be a sinusoid with the frequency ω_0.

Very intuitive idea: It seems reasonable that an oscillation can be maintained if the signal at B has the same amplitude and phase as the injected signal!

Critical point: -1



(a) Closed loop system (b) Open loop system

- Tracing signals around the loop, we find that the signals at A and B are identical if there is a frequency ω_0 such that

$$L(i\omega_0) = -1. \tag{1}$$

- This provides a condition for maintaining an **oscillation**.
- The condition (1) implies that the frequency response goes through the value -1, which is called the **critical point**.

Letting ω_c represent a frequency at which $\angle L(i\omega_c) = 180^\circ$,

- we can further reason that the system is stable if $|L(i\omega_c)| < 1$, since the signal at point B will have smaller amplitude than the injected signal.
- A rigorous version is the **Nyquist's stability criterion**.

Example: Electric motor

Example

- The process dynamics are $P(s) = \frac{k_I}{Js^2 + cs}$.
- We use a proportional controller $C(s) = k_{\mathrm{p}}$.
- The loop transfer function for the control system is

$$L(s) = P(s)C(s)e^{-\tau s} = \frac{k_I k_p}{Js^2 + cs} e^{-\tau s},$$

where τ is the delay in sensing of the motor position.

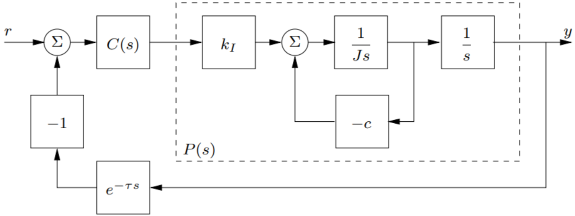
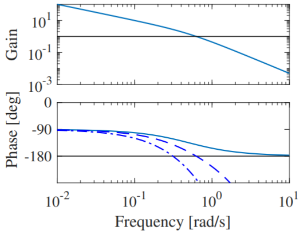
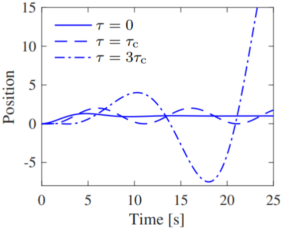


Figure: Block diagram of motor control with a short delay in the sensed position of the motor.

Example: Electric motor



(a) Loop transfer function



(b) Closed loop step responses

Figure: Loop transfer function and step response for the DC motor control system. The system parameters are $k_I = 1$, $J = 2$, $c = 1$ and the controller parameters are $k_p = 1$ and $\tau = 0, 1$, and 3.

- **Nyquist's approach**: it allows us to study the stability of the feedback system by looking at properties of the loop transfer function $L = PC$.
- More generally: how the controller should be chosen to obtain a desired loop transfer function.
- Different ways for this, called **loop shaping**, will be discussed in Week 8/9.

Outline

Nyquist[1] plot

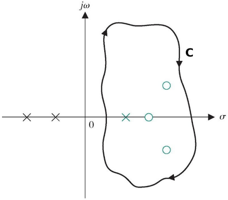
- Frequency response of an LTI system: **Bode plot** of its transfer function
- Stability of a closed-loop system: **Nyquist plot** of its loop transfer function

H. Nyquist (1889 – 1976)

Definition (Nyquist plot)

The **Nyquist plot** of the loop transfer function $L(s)$ is the image of $L(s)$ by tracing $s \in \mathbb{C}$ around the **Nyquist contour**.

- A **contour** is a piecewise smooth path in the complex plane
- A contour is **closed** if it starts and ends at the same point
- A contour is **simple** if it does not cross itself at any point



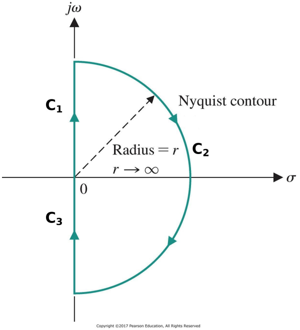
Nyquist's stability criterion utilizes **contours** in the complex plane to relate the **locations** of the open-loop and closed-loop poles.

[1] Harry Nyquist; `https://en.wikipedia.org/wiki/Harry_Nyquist`

Nyquist contour

The (standard or simplest) Nyquist contour, also known as "Nyquist D contour" ($\Gamma \subset \mathbb{C}$), is made up of three parts:

- **Contour** C_1: points $s = i\omega$ on the positive imaginary axis, as ω ranges from 0 to ∞
- **Contour** C_2: points $s = Re^{i\theta}$ on a semi-circle as $R \to \infty$ and θ ranges from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$
- **Contour** C_3: points $s = i\omega$ on the negative imaginary axis, as ω ranges from $-\infty$ to 0



The image of $L(s)$ when s traverses Γ gives a closed curve in the complex plane and is referred to as the **Nyquist plot** for $L(s)$.

Example 1: a third-order system

Draw a Nyquist plot for $L(s) = \frac{1}{(s+a)^3}$.

- **Counter** C_1: $s = i\omega$ with ω from 0 to ∞

$$L(i0) = \frac{1}{a^3}\angle 0^\circ, \qquad L(i\infty) = 0\angle -270^\circ$$

- for $0 < \omega < \infty$

$$L(i\omega) = \frac{1}{(i\omega + a)^3} = \frac{(a - i\omega)^3}{(a^2+\omega^2)^3} = \frac{a^3 - 3a\omega^2}{(a^2+\omega^2)^3} + i\frac{\omega^3 - 3a^2\omega}{(a^2+\omega^2)^3}$$

- **Counter** C_2: $s = Re^{i\theta}$ for $R \to \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$.

$$L(Re^{i\theta}) = \frac{1}{(Re^{i\theta} + a)^3} \to 0$$

- **Counter** C_3: $s = i\omega$ with $\omega \in (-\infty, 0)$

$$L(-i\omega) = L(\bar{i}\omega) = \overline{L(i\omega)}$$

which is a *reflection* (complex conjugate) of $L(C_1)$ about the real axis.

Example 1: a third-order system

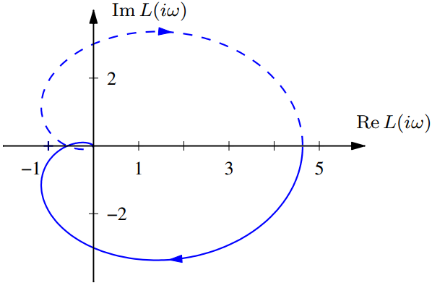


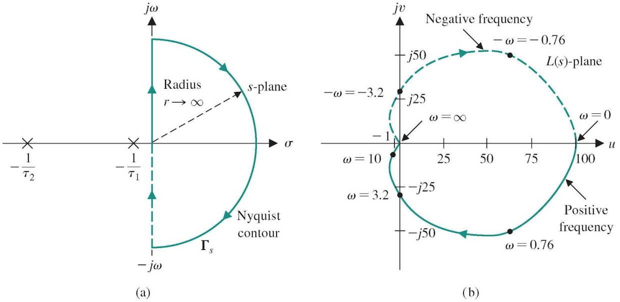
Figure 10.5: Nyquist plot for a third-order transfer function $L(s)$. The Nyquist plot consists of a trace of the loop transfer function $L(s) = 1/(s+a)^3$ with $a = 0.6$. The solid line represents the portion of the transfer function along the positive imaginary axis, and the dashed line the negative imaginary axis. The outer arc of the Nyquist contour Γ maps to the origin.

Example 2: a second-order system

Draw a Nyquist plot for

$$L(s) = \frac{100}{(1+s)(1+s/10)}.$$

- **Contour** C_1: $L(i0) = 100\angle 0^\circ$, $L(i\infty) = 0\angle -180^\circ$
- **Contour** C_2: $\lim_{R\to\infty} L(Re^{i\theta}) = 0$



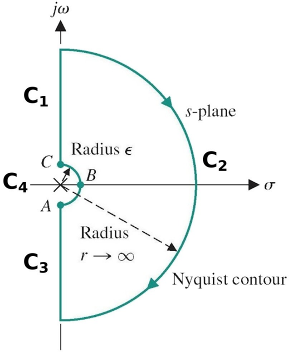
Pole/Zero on the Imaginary Axis

- When the loop transfer function has poles on the imaginary axis, the gain is infinite at the poles.
- The Nyquist contour needs to be modified to take a small detour around such poles or zeros
- So, we add another part: **Contour** C_4
  - plot $L(\epsilon e^{i\theta})$ for $\epsilon \to 0$ and

$$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

  - substitute $s = \epsilon e^{i\theta}$ into $L(s)$ and examine what happens as

$$\epsilon \to 0$$

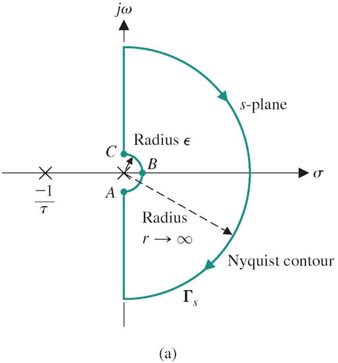


Example 3

Draw a Nyquist plot for a loop transfer system:

$$L(s) = \frac{\kappa}{s(1+\tau s)}$$

- Since there is a pole at the origin, we need to use a modified Nyquist contour



Example 3

- **Contour** C_4 with $s = \epsilon e^{i\theta}$ for $\epsilon \to 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$:

$$\lim_{\epsilon \to 0} L(\epsilon e^{i\theta}) = \lim_{\epsilon \to 0} \frac{\kappa}{\epsilon e^{i\theta}} = \lim_{\epsilon \to 0} \frac{\kappa}{\epsilon} e^{-i\theta} = \infty \angle -\theta$$

  - The phase of $L(s)$ changes from $\dfrac{\pi}{2}$ at $\omega = 0^-$ to $-\dfrac{\pi}{2}$ at $\omega = 0^+$
- **Contour** C_1 with $\omega \in (0, \infty)$:

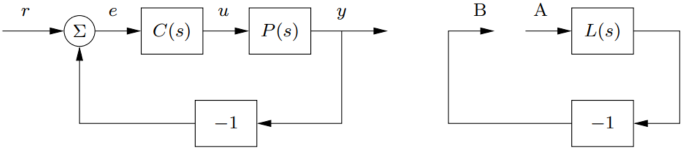
$$\begin{aligned} L(i0^+) &= \infty \angle -90^\circ \\ L(i\infty) &= \lim_{\omega \to \infty} \frac{\kappa}{i\omega(1 + i\omega\tau)} = \lim_{\omega \to \infty} \left| \frac{\kappa}{\tau\omega^2} \right| \angle -\pi/2 - \tan^{-1}(\omega\tau) \\ &= 0 \angle -180^\circ \end{aligned}$$

- **Contour** C_2 with $s = re^{i\theta}$ for $r \to \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$:

$$\lim_{r \to \infty} L(re^{i\theta}) = \lim_{r \to \infty} \left| \frac{\kappa}{\tau r^2} \right| e^{-2i\theta} = 0 \angle -2\theta$$

  - The phase of $L(s)$ changes from $-\pi$ at $\omega = \infty$ to π at $\omega = -\infty$
- **Contour** C_3 with $\omega \in (-\infty, 0)$:
  - $L(C_3)$ is a **reflection** of $L(C_1)$ about the real axis

Simplified Nyquist Criterion



(a) Closed loop system (b) Open loop system

Theorem (Simplified Nyquist Criterion)

Let $L(s)$ be the loop transfer function for a negative feedback system, and assume that L has no poles in the closed right half-plane ($\mathrm{Re}(s) \geq 0$) except possibly at the origin. Then the closed loop system

$$G_{\mathrm{cl}}(s) = \frac{L(s)}{1 + L(s)}$$

is stable if and only if the image of $L(s)$ along the closed contour Γ has no net encirclements of the critical point $s = -1$.

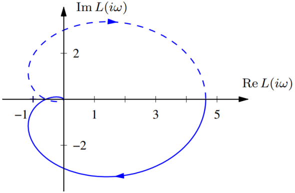
Winding number

The following conceptual procedure can be used to determine that there are no net encirclements.

- Step 1: Fix a pin at the critical point $s = -1$, orthogonal to the plane.
- Step 2: Attach a string with one end at the critical point and the other on the Nyquist plot.
- Step 3: Let the end of the string attached to the Nyquist curve traverse the whole curve.

There are no encirclements if the string does not wind up on the pin when the curve is encircled.

- The number of encirclements is called the **winding number**.



Nyquist plot for $L(s) = \frac{1}{(s+a)^3}$ with $a = 0.6$

- Closed-loop system

$$G_{\text{cl}}(s) = \frac{L(s)}{1 + L(s)} = \frac{1}{(s+0.6)^3 + 1}, \qquad \lambda_1 = -1.6000, \lambda_{2,3} = -0.1 \pm 0.8660i$$

Outline

Loop Transfer Function

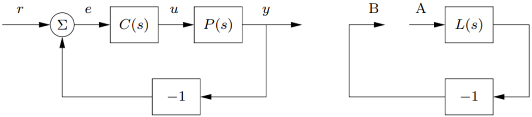
Nyquist plot and The Nyquist Criterion

Summary

Summary

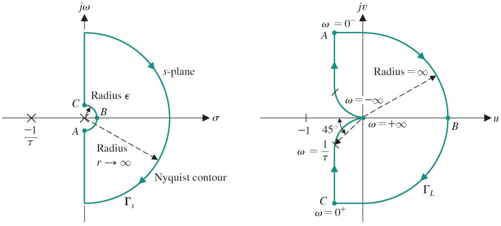
- Nyquist's idea was to first investigate conditions under which oscillations can occur in a feedback loop – The **Loop transfer function**:

$$L(s) = P(s)C(s).$$



(a) Closed loop system (b) Open loop system

- **Nyquist plot** and Simplified **Nyquist criterion**



(a) (b)