ECE 171A: Linear Control System Theory
Lecture 17: Nyquist Criterion and Stability margins

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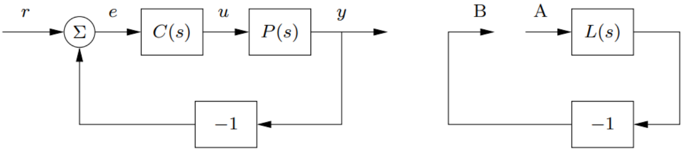
Reading materials: Ch 10.3, 10.4

Outline

Simplified Nyquist Criterion



(a) Closed loop system (b) Open loop system

Theorem (Simplified Nyquist Criterion)

Let $L(s)$ be the loop transfer function for a negative feedback system, and assume that L has no poles in the closed right half-plane ($\operatorname{Re}(s) \geq 0$) except possibly at the origin. Then the closed loop system

$$G_{\text{cl}}(s) = \frac{L(s)}{1 + L(s)}$$

is stable if and only if the image of $L(s)$ along the closed contour Γ has no net encirclements of the critical point $s = -1 + i0$.

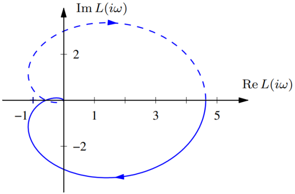
Winding number

The following conceptual procedure can be used to determine that there are no net encirclements.

- Step 1: Fix a pin at the critical point $s = -1$, orthogonal to the plane.
- Step 2: Attach a string with one end at the critical point and the other on the Nyquist plot.
- Step 3: Let the end of the string attached to the Nyquist curve traverse the whole curve.

There are no net encirclements if the string does not wind up on the pin when the curve is encircled.

- The number of net encirclements is called the **winding number**.



Nyquist plot for $L(s) = \frac{1}{(s+a)^3}$ with $a = 0.6$

- Closed-loop system

$$G_{\text{cl}}(s) = \frac{L(s)}{1 + L(s)} = \frac{1}{(s + 0.6)^3 + 1}, \qquad \lambda_1 = -1.6000, \lambda_{2,3} = -0.1 \pm 0.8660i$$

Outline

Cauchy's Principle of the Argument

Nyquist Stability Criterion

Stability margins

Summary

Loop Transfer Function

Loop transfer function

$$L(s) = P(s)C(s) \qquad \Rightarrow \qquad G_{\text{cl}}(s) = \frac{L(s)}{1 + L(s)}.$$

- Consider a control system with a loop transfer function:

$$L(s) = \kappa \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}$$

- At each s, $L(s)$ is a complex number with magnitude and phase:

$$|L(s)| = |\kappa| \frac{\prod_{i=1}^{m} |s - z_i|}{\prod_{i=1}^{n} |s - p_i|} \qquad \angle L(s) = \angle \kappa + \sum_{i=1}^{m} \angle(s - z_i) - \sum_{i=1}^{n} \angle(s - p_i)$$

- Graphical evaluation of the magnitude and phase:
  - $|s - z_i|$ is the length of the vector from z_i to s
  - $|s - p_i|$ is the length of the vector from p_i to s
  - $\angle(s - z_i)$ is the angle from the real axis to the vector from z_i to s
  - $\angle(s - p_i)$ is the angle from the real axis to the vector from p_i to s

Evaluating $L(s)$ along a Contour

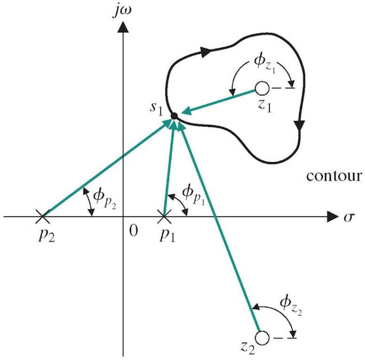
Let C be a simple closed clockwise contour in the complex plane; Evaluating $L(s)$ at all points on C produces a new closed contour $L(C)$

— **image of C under $L(s)$**.

Assumption: C does not pass through the origin or any of the poles or zeros of $L(s)$ (otherwise $\angle L(s)$ is undefined). **Effects of poles and zeros**:

- A zero z_i outside the contour C:
  - As s moves around the contour C, the vector $s - z_i$ swings up and down but not all the way around
  - Thus, the net change in $\angle(s - z_i)$ is 0
- A zero z_i inside the contour C:
  - As s moves around the contour C, the vector $s - z_i$ turns all the way around
  - Thus, the net change in $\angle(s - z_i)$ is -2π
- A pole p_i outside the contour C: the net change in $\angle(s - p_i)$ is 0
- A pole p_i inside the contour C: the net change in $\angle(s - p_i)$ is -2π

Evaluating $L(s)$ along a Contour



- A zero z_i outside the contour C: The net change in $\angle(s - z_i)$ is 0
- A zero z_i inside the contour C: The net change in $\angle(s - z_i)$ is -2π
- A pole p_i outside the contour C: the net change in $\angle(s - p_i)$ is 0
- A pole p_i inside the contour C: the net change in $\angle(s - p_i)$ is -2π

Principle of the Argument

- Let Z and P be the number of zeros and poles of $L(s)$ inside C
- As s moves around C, $\angle L(s)$ undergoes a net change of $-(Z-P)2\pi$
- A net change of -2π means that the vector from 0 to $L(s)$ swings **clockwise** around the origin one full rotation
- A net change of $-(Z-P)2\pi$ means that the vector from 0 to $L(s)$ must encircle the origin in **clockwise** direction $(Z-P)$ times

Theorem (Cauchy's Principle of the Argument)

Consider a transfer function $L(s)$ and a simple closed clockwise contour C. Let Z and P be the number of zeros and poles of $L(s)$ inside C.

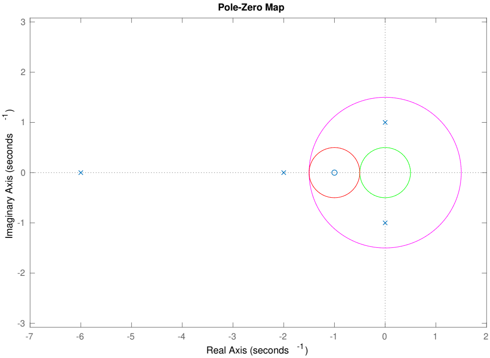
- *Then, the contour generated by evaluating $L(s)$ along C will encircle the origin in a clockwise direction $Z-P$ times.*

- Note that Cauchy's Principle of the Argument works for **any transfer function** — $L(s)$ above does not need to be a loop transfer function.

Principle of the Argument: Example

- Pole-zero map for

$$G(s) = \frac{10(s+1)}{(s+2)(s^2+1)(s+6)}$$



Principle of the Argument: Example

- A circle contour C centered at the origin with radius 0.5 (green)
- The contour may be parameterized by $z(t) = 0.5e^{-it}$ for $t \in [0, 2\pi]$
- The contour C is mapped by $G(s)$ to a new contour (from blue to red), e.g., parameterized by $G(z(t))$ for $t \in [0, 2\pi]$

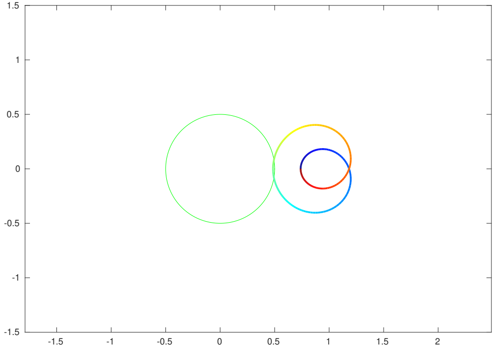


Figure: Encircle the origin in a clockwise direction $Z - P = 0$ times

Principle of the Argument: Example

- A circle contour C centered at $(-1, 0)$ with radius 1 (red)
- The contour C is mapped by $G(s)$ to a new contour (from blue to red)

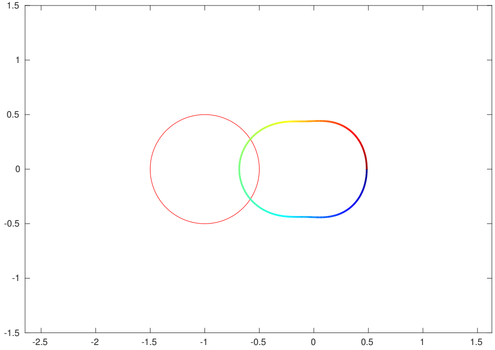


Figure: Encircle the origin in a clockwise direction $Z - P = 1$ time

Principle of the Argument: Example

- A circle contour C centered at the origin with radius 1.5 (magenta)
- The contour C is mapped by $G(s)$ to a new contour (from blue to red)

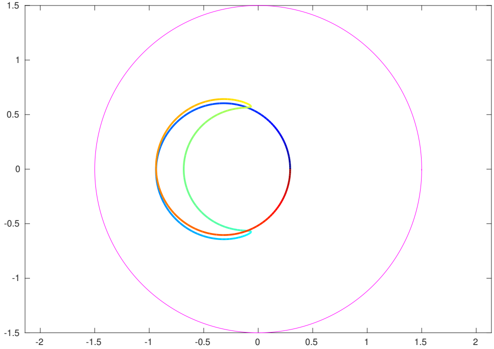


Figure: Encircle the origin in a clockwise direction $Z - P = 1 - 2 = -1$ time

Outline

Nyquist Stability Criterion

- Consider the stability of the closed-loop transfer function:

$$G_{\mathrm{cl}}(s) = \frac{L(s)}{1 + L(s)} = \frac{L(s)}{\Delta(s)}$$

- The **poles** of $\Delta(s)$ are the poles of $L(s)$ — **open-loop poles**
- The **zeros** of $\Delta(s)$ are the poles of $G_{\mathrm{cl}}(s)$ — **closed-loop poles**
- Principle of the Argument applied to $\Delta(s) = 1 + L(s)$:
  - Let Γ be a Nyquist contour.
  - Let Z be the number of zeros of $\Delta(s)$ (closed-loop poles) inside Γ.
  - Let P be the number of poles of $\Delta(s)$ (open-loop poles) inside Γ.
  - Then, the image of Γ under $\Delta(s)$, denoted as $\Delta(\Gamma)$, encircles the origin in clockwise direction $N = Z - P$ times.

Nyquist Stability Criterion

- From the Principle of the Argument applied to $\Delta(s)$, the number of closed-loop poles in the closed right half-plane is:

$$Z = N + P$$

  - N: the **clockwise encirclements of the origin** by $\Delta(\Gamma)$ correspond to the **clockwise encirclements of** $-1 + i0$ by $L(\Gamma)$ and can be determined from a **Nyquist plot** of $L(s)$
  - P: the number of poles of $\Delta(s)$ inside C corresponds to the number of poles of $L(s)$ inside Γ and can be determined from $L(s)$ or its Bode plot

Theorem (Nyquist Stability Criterion)

Consider a unity feedback control system with open-loop transfer function $L(s)$. Let Γ be a Nyquist contour. The system is stable if and only if **the number of counterclockwise encirclements** *of $-1 + i0$ by the Nyquist plot $L(\Gamma)$ is equal to* **the number of poles** *of $L(s)$ inside Γ.*

Nyquist Stability Criterion

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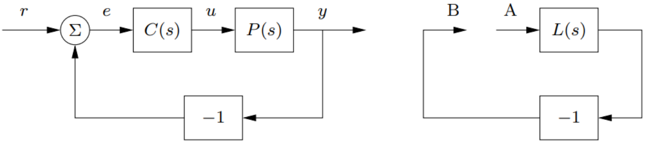
Consider a unity feedback control system with open-loop transfer function $L(s)$. Let Γ be a Nyquist contour. The system is stable if and only if the number of counterclockwise encirclements of $-1 + i0$ by the Nyquist plot $L(\Gamma)$ is equal to the number of poles of $L(s)$ inside Γ.



(a) Closed loop system (b) Open loop system

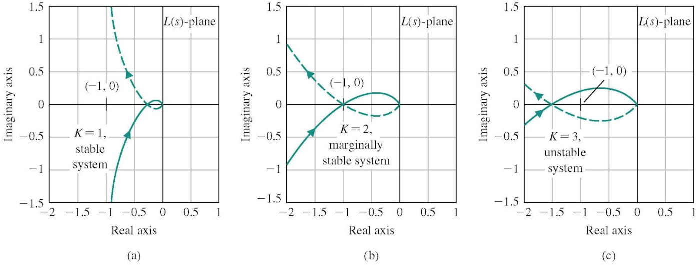
Theorem (Simplified Nyquist Criterion)

Let $L(s)$ be the loop transfer function for a negative feedback system, and assume that L has no poles in the closed right half-plane ($\mathrm{Re}(s) \geq 0$) except possibly at the origin. Then the closed loop system is stable if and only if the image of $L(s)$ along the closed contour Γ has no net encirclements of the critical point $s = -1 + i0$.

Nyquist Stability: Example

Determine the closed-loop stability of

$$L(s) = \frac{\kappa}{s(1+\tau_1 s)(1+\tau_2 s)} = \frac{\kappa}{s(1+s)^2}$$



- $G(C_1)$ crosses the real axis when:

$$G(i\omega) = \frac{-\kappa(\tau_1+\tau_2) - i\kappa(1-\omega^2\tau_1\tau_2)\omega}{1+\omega^2(\tau_1^2+\tau_2^2)+\omega^4\tau_1^2\tau_2^2} = \alpha + i0$$

$$\Rightarrow \omega = \frac{1}{\sqrt{\tau_1\tau_2}} \qquad \alpha = -\frac{\kappa\tau_1\tau_2}{\tau_1+\tau_2}$$

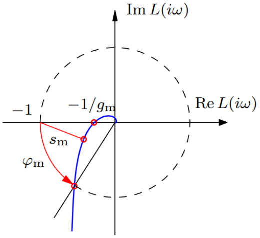
- The system is stable when $\alpha = -\frac{\kappa\tau_1\tau_2}{\tau_1+\tau_2} > -1 \quad \Leftrightarrow \quad \kappa < \frac{\tau_1+\tau_2}{\tau_1\tau_2} = 2.$

Outline

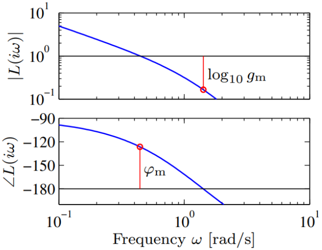
Stability Margin

In practice, it is not enough that a system is stable. There must also be some margins of stability that describe how far from instability the system is and its **robustness to perturbation**.

- **Stability margins** express how well the Nyquist curve of the loop transfer avoids the critical point -1.
- The shortest distance s_m of the Nyquist curve to the critical point is a natural criterion — **stability margin**.

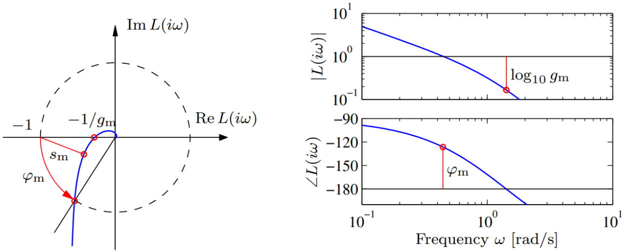


(a) Nyquist plot



(b) Bode plot

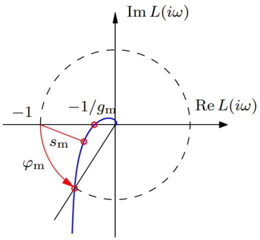
Gain Margin



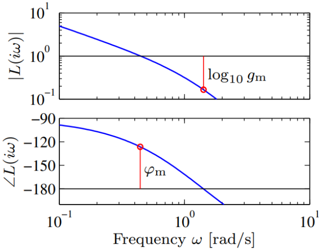
(a) Nyquist plot (b) Bode plot

- **Gain Margin**:
  - the factor by which the open-loop gain can be increased before a stable closed-loop system becomes unstable
  - It is the **inverse** of the distance between the origin and the point between -1 and 0 where the loop transfer function crosses the negative real axis.
  - On a Nyquist plot, the gain margin is the **inverse** of the distance to the first point where $G(C)$ crosses the real axis.

Phase Margin



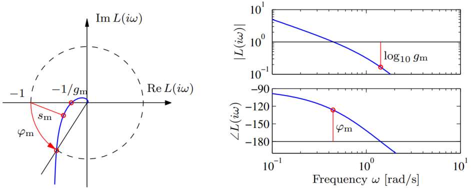
(a) Nyquist plot



(b) Bode plot

- **Phase Margin**:
  - the amount by which the open-loop phase can be decreased before a stable closed-loop system becomes unstable
  - i.e. the amount of phase lag required to reach the stability limit
- On a Nyquist plot, the phase margin is the smallest angle on the unit circle between -1 and $G(C)$

Algebraic Definitions



(a) Nyquist plot (b) Bode plot

- **Phase-Crossover frequency**
  - $\omega_{\rm pc}$ at which $L(i\omega)$ crosses the real axis: $\angle L(i\omega_{\rm pc}) = -180^\circ$
- **Gain Margin**
  - the inverse of the open-loop gain at $\omega_{\rm pc}$: $g_{\rm m} = \frac{1}{|L(i\omega_{\rm pc})|}$
- **Gain-Crossover frequency**
  - $\omega_{\rm gc}$ at which $G(j\omega)$ crosses the unit circle: $|L(i\omega_{\rm gc})| = 1$
- **Phase Margin**
  - the amount by which the open-loop phase at ω_g exceeds -180°:

$$\varphi_{\rm m} = \angle L(i\omega_{\rm gc}) + 180^\circ$$

Stability margins for a third-order system

Example

Consider a loop transfer function $L(s) = \dfrac{3}{(s+1)^3}$

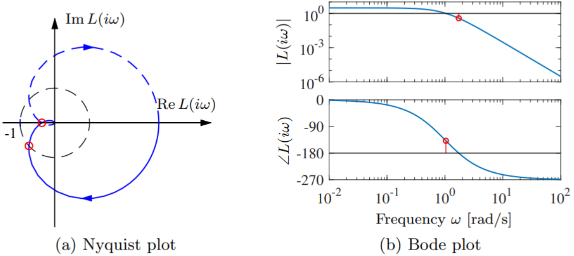


Figure: Stability margins for a third-order transfer function. (a) Nyquist plot; (b) Bode plot.

- We can use its Nyquist plot or Bode plot. This yields the following values:

$$g_{\mathrm{m}} = 2.67, \qquad \varphi_{\mathrm{m}} = 41.7^\circ, \qquad s_{\mathrm{m}} = 0.464.$$

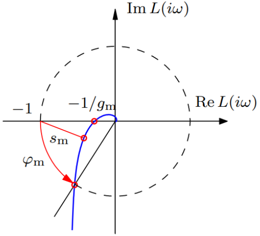
Outline

Summary

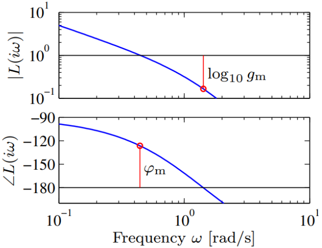
Theorem (Nyquist's Stability Criterion)

Consider a unity feedback control system with open-loop transfer function $L(s)$. Let Γ be a Nyquist contour. The system is stable if and only if the number of counterclockwise encirclements of $-1 + i0$ by the Nyquist plot $L(\Gamma)$ is equal to the number of poles of $L(s)$ inside Γ.

Classical robustness measures: stability margin, phase margin, gain margin



(a) Nyquist plot



(b) Bode plot