ECE 171A: Linear Control System Theory
Lecture 18: Bode's relations and Root locus

Yang Zheng

Assistant Professor, ECE, UCSD

May 09, 2022

Reading materials: Ch 10.4, Ch 12.5

Something NOT fun

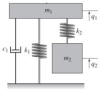


Figure 3: A vibration damper. Vibrations of the mass m_1 can be damped by providing it with an auxiliary mass m_2, attached to m_1 by a spring with stiffness k_2. The parameters m_2 and k_2 are chosen so that the frequency $\sqrt{k_2/m_2}$ matches the frequency of the vibration.

(c) We let $m_1 = 1; c_1 = 1; k_1 = 1; m_2 = 1; k_2 = 1$. Draw the bode plot of $G_{q_1F}(s)$ in Matlab. Then, simulate the response $q_1(t)$ when we apply inputs $F(t) = \sin(\omega t)$ with $\omega = 0.0.578, 1$, and 1.1 for 100 seconds (the initial state is $q_1(0) = 0, q_2(0) = 0, \dot{q}_1(0) = 0, \dot{q}_2(0) = 0.)^6$. Are these responses consistent with the bode plot? Discuss the blocking property of the zeros in $G_{q_1F}(s)$. [10 pts]

Show transcribed image text

Expert Answer

This problem has been solved!

See the answer

- Not fun, and not necessary! Q4 in HW5 will not be graded (Max 70).
- Next time, I will refer it to the Office of Student Conduct for investigation.
- All of us should uphold academic integrity — more important than grades.

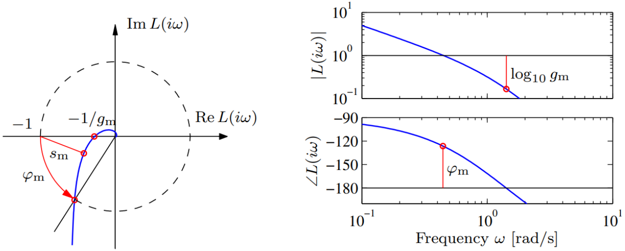
Outline

Nyquist's Stability Criterion

Theorem (Nyquist's Stability Criterion)

Consider a unity feedback control system with open-loop transfer function $L(s)$. Let Γ be a Nyquist contour. The system is stable if and only if the number of counterclockwise encirclements of $-1 + i0$ by the Nyquist plot $L(\Gamma)$ is equal to the number of poles of $L(s)$ inside Γ.

Classical robustness measures: stability margin, phase margin, gain margin



(a) Nyquist plot (b) Bode plot

Outline

Bode's relations

From Bode plots, there appears to be a relation between the gain curve and the phase cure — **Minimum phase systems**

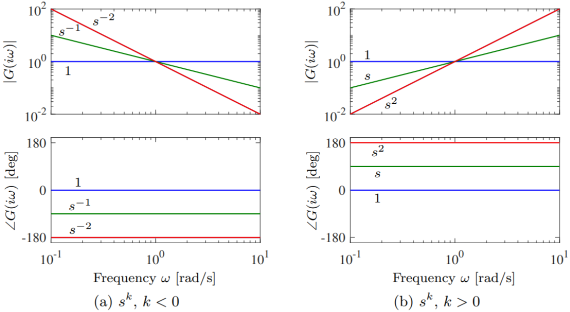


Figure: Bode plots of the transfer functions $P(s) = s^k$ for $k = -2, -1, 0, 1, 2$.

- **Differentiator** s: the slope is $+1$, and the phase is a constant 90°;
- **Integrator** $\frac{1}{s}$: the slope is -1, and the phase is a constant -90°.
- **First-order system** $s + a$: slop 0 for small frequencies, and slop $+1$ for high frequencies; Phase 0 for low frequencies and 90° for high frequencies.

Minimum phase systems

Minimum phase systems: they have the smallest phase lag of all systems with the same gain curve.

- No time delays or poles and zeros in the right half-plane.
- Have the property that $\log|P(s)|/s \to 0$ as $s \to \infty$ for $\mathrm{Re}(s) \geq 0$.

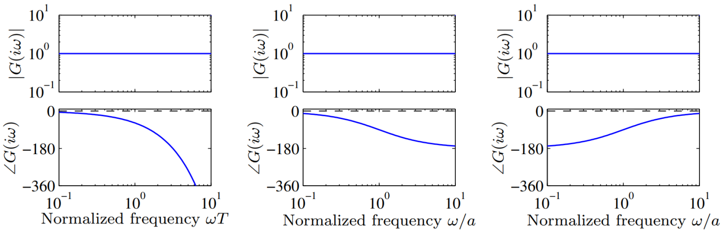
For minimum phase systems, the phase is uniquely given by the shape of the gain curve and vice versa:

$$\arg G(i\omega_0) = \frac{\pi}{2} \int_0^\infty f(\omega) \frac{d \log|G(i\omega)|}{d \log \omega} \frac{d\omega}{\omega},$$

where f is the weighting kernel

$$f(\omega) = \frac{2}{\pi^2} \log \left| \frac{\omega + \omega_0}{\omega - \omega_0} \right| \qquad \text{and} \qquad \int_0^\infty f(\omega) \frac{d\omega}{\omega} = 1.$$

Non-minimum phase systems



(a) Time delay (b) Right half-plane zero (c) Right half-plane pole

Figure: Bode plots of systems that are not minimum phase. (a) Time delay $P(s) = e^{-sT}$, (b) system with a right half-plane (RHP) zero $P(s) = (a - s)/(a + s)$, and (c) system with right half-plane pole $P(s) = (s + a)/(s - a)$. The corresponding minimum phase system has the transfer function $P(s) = 1$ in all cases.

- The presence of poles and zeros in the right half-plane imposes **severe limits on the achievable performance** — Week 10
- The **poles** are intrinsic properties of the system and they do not depend on sensors and actuators.
- The **zeros** depend on how inputs and outputs of a system are coupled to the states. They can be changed by moving/adding sensors and actuators.

Outline

Root locus - Overview

Motivation: System responses are affected by the locations of the poles of its transfer function in the complex domain, e.g., *stability*, *convergence speed*, etc.

- Feedback control can move the closed-loop system poles by designing an appropriate controller – **pole placement** (not covered in this course).

What is the root locus method?

- The **root locus** is a graph of the roots of the characteristic polynomial as a function of a parameter — give insight into the effects of the parameter.
- i.e., the **root locus** provides all possible pole locations as a system parameter (e.g., the controller gain) varies
- **Obtain the root locus** — find the roots of the closed loop characteristic polynomial for different values of the parameter (*easy for computers*).
- The general shape of the root locus can be obtained with very **little computational effort**, and that it often gives *considerable insight*.

Root locus: Example 1

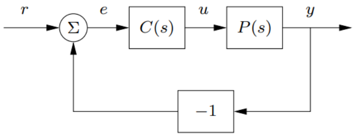


Figure: Feedback control system

- Consider a single-loop feedback control system with

$$P(s) = \frac{1}{s(s+2)}, \qquad C(s) = k$$

- The closed-loop transfer function from the reference r to output y is:

$$G_{\mathrm{yr}}(s) = \frac{kP(s)}{1+kP(s)} = \frac{k}{s^2+2s+k}$$

- How do the closed-loop poles vary as a function of k?
  - We can actually compute the roots as $\lambda_{1,2} = -1 \pm \sqrt{1-k}$.

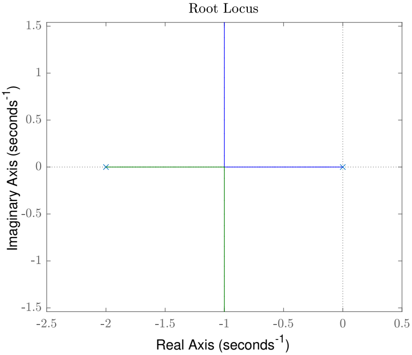
Root Locus: Example 1

Example

- Root locus for

$$P(s) = \frac{1}{s(s+2)}$$

- Matlab command: `rlocus(tf([1],[1 2 0]))`.



Root Locus: Example 2

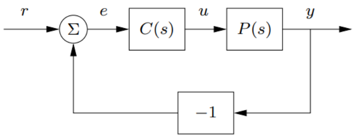


Figure: Feedback control system

Example

- Consider a single-loop feedback control system with

$$P(s) = \frac{(s+3)}{s(s+2)}, \qquad C(s) = k$$

- The closed-loop transfer function from r to y is:

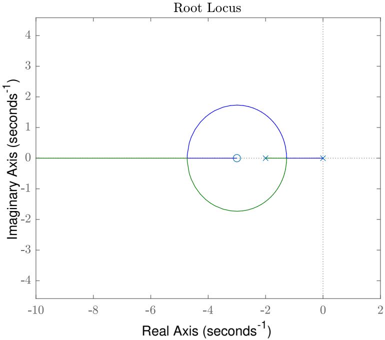
$$G_{\text{yr}}(s) = \frac{kP(s)}{1+kP(s)} = \frac{k(s+3)}{s^2+(2+k)s+3k}$$

Root Locus: Example 2

- Root locus for

$$P(s) = \frac{(s+3)}{s(s+2)}$$

- Matlab command: `rlocus(tf([1 3],[1 2 0]))`.



- In this case, adding a stable zero in the open-loop system increases the relative stability of the closed-loop system by attracting the branches of the root locus.

Root Locus: Example 3

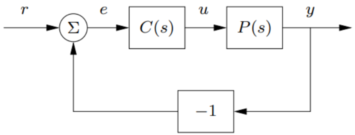


Figure: Feedback control system

Example

- Consider a single-loop feedback control system with

$$P(s) = \frac{1}{s(s+2)(s+3)}, \qquad C(s) = k$$

- The closed-loop transfer function from r to y is:

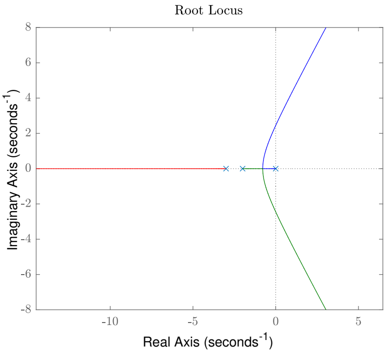
$$G_{\mathrm{yr}}(s) = \frac{k}{s^3 + 5s^2 + 6s + k}$$

Root Locus: Example 3

- Root locus for

$$P(s) = \frac{1}{s(s+2)(s+3)}$$

- Matlab command: `rlocus(tf([1],[1 5 6 0]))`.



- In this case, adding a stable pole in the open-loop system makes the closed-loop system less stable (stable for some values of k);

Root Locus Definition

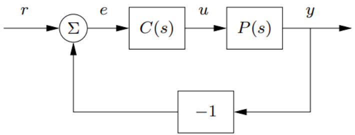


Figure: Feedback control system

- Closed-loop transfer function:

$$G_{\mathrm{yr}}(s) = \frac{kP(s)}{1 + kP(s)}$$

- The closed-loop poles satisfy:

$$1 + kP(s) = 0$$

- The **root locus** is the set of points s such that $1 + kP(s) = 0$ as k varies

Root Locus Definition

Consider the zeros and poles of $P(s)$ explicitly:

$$P(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$
$$= b_m \frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}$$

- The closed loop characteristic polynomial is:

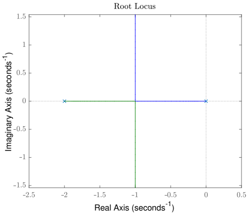
$$1 + kP(s) = 0 \qquad \Rightarrow \qquad a_{\text{cl}}(s) := a(s) + kb(s) = 0$$

- The closed loop poles are the roots of $a_{\text{cl}}(s)$.
- The **root locus** is a graph of the roots of $a_{\text{cl}}(s)$ as the gain k is varied from 0 to ∞.
- Since the polynomial $a_{\text{cl}}(s)$ has degree n, the plot will have n branches.

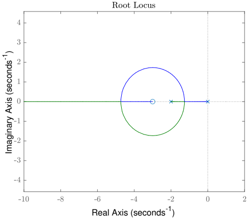
Starting and ending points of Root locus

- Each branch starts at a different open-loop pole.
- m of the branches end at different open-loop zeros.
- The remaining $n - m$ branches go to infinity

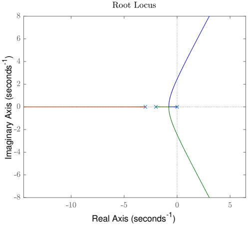
Example



(a) $P(s) = \frac{1}{s(s+2)}$



(b) $P(s) = \frac{s+3}{s(s+2)}$



(c) $P(s) = \frac{1}{s(s+2)(s+3)}$

Starting and ending points of Root locus

- The closed loop characteristic polynomial is:

$$1 + kP(s) = 0 \qquad \Rightarrow \qquad a_{\text{cl}}(s) := a(s) + kb(s) = 0$$

- The **root locus** is a graph of the roots of $a_{\text{cl}}(s)$ as the gain k is varied from 0 to ∞.

Starting points when $k = 0$: we have $a_{\text{cl}}(s) := a(s) + kb(s) = a(s)$.

- The closed-loop poles are equal to the open-loop poles.
- Open-loop poles at $s = p$ with multiplicity $l \Rightarrow$

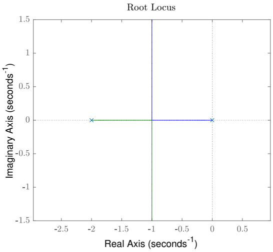
$$a(s) + kb(s) = (s - p)^l \tilde{a}(s) + kb(s) \approx (s - p)^l \tilde{a}(p) + kb(p) = 0$$

For **small value** of k, we have the roots are

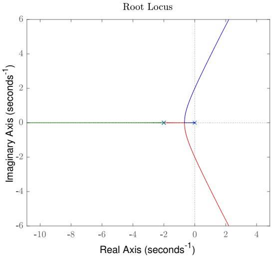
$$s = p + \sqrt[l]{-kb(p)/\tilde{a}(p)}$$

- The root locus has a **star pattern** with l branches from the open-loop pole $s = p$, and the angle between two neighboring branches is $\frac{2\pi}{l}$.

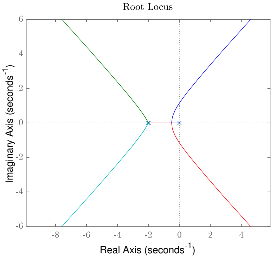
Examples



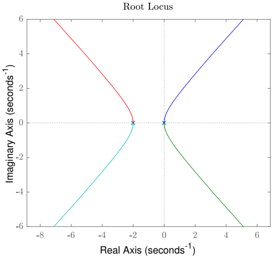
(a) $P(s) = \frac{1}{s(s+2)}$



(b) $P(s) = \frac{1}{s(s+2)^2}$



(c) $P(s) = \frac{1}{s(s+2)^3}$



(d) $P(s) = \frac{1}{s^2(s+2)^2}$

Starting and ending points of Root locus

- The closed loop characteristic polynomial is:

$$1 + kP(s) = 0 \qquad \Rightarrow \qquad a_{\text{cl}}(s) := a(s) + kb(s) = 0$$

- The **root locus** is a graph of the roots of $a_{\text{cl}}(s)$ as the gain k is varied from 0 to ∞.

Ending points when k goes to infinity: we have

$$a_{\text{cl}}(s) := b(s)\left(\frac{a(s)}{b(s)} + k\right) \approx b(s)\left(\frac{s^{n-m}}{b_0} + k\right)$$

- For large K, the **closed-loop poles** are approximately the **roots (zeros of $P(s)$)** of $b(s)$ and

$$\sqrt[n-m]{-b_0 k}$$

- A better approximation of the **closed-loop poles** is

$$s = s_0 + \sqrt[n-m]{-b_0 k}, \quad s_0 = \frac{1}{n-m}\left(\sum_{k=1}^{n} p_k - \sum_{k=1}^{m} z_k\right).$$

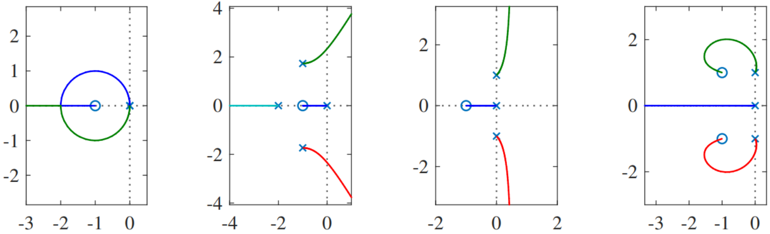
Examples

Example

Show the root loci for the following open-loop transfer functions

$$P_a(s) = \frac{s+1}{s^2}, \qquad P_b(s) = \frac{s+1}{s(s+2)(s^2+2s+4)},$$

$$P_c(s) = \frac{s+1}{s(s^2+1)}, \quad P_d(s) = \frac{s^2+2s+2}{s(s^2+1)}.$$



(a) $P_a(s)$ (b) $P_b(s)$ (c) $P_c(s)$ (d) $P_d(s)$

Outline

Summary

- **Minimum phase systems**: they have the smallest phase lag of all systems with the same gain curve — for these systems
  - No time delays or poles and zeros in the right half-plane.
  - Have the property that $\log|P(s)|/s \to 0$ as $s \to \infty$ for $\mathrm{Re}(s) \geq 0$.
- For minimum phase systems, the phase is uniquely given by the shape of the gain curve and vice versa
- **Root locus**: a graph of the roots of $a_{\mathrm{cl}}(s)$ as the gain k is varied from 0 to ∞.
  - The plot of root locus will have n branches.
  - Each branch starts at a different open-loop pole.
  - m of the branches end at different open-loop zeros.
  - The remaining $n - m$ branches go to infinity.