# ECE 171A: Linear Control System Theory
# Lecture 19: PID control (I)


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Reading materials: Ch 11.1, 11.2

# Outline

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# Overview

**Proportional-integral-derivative (PID) control** is by far the most common way of using feedback in engineering systems

- A survey of controllers for more than 100 boiler-turbine units: 94.4% of all controllers were PI, 3.7% PID, and 1.9% used advanced control.

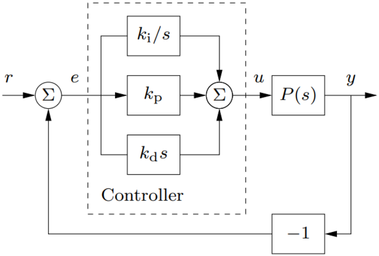



Figure: PID using error feedback

PID control

- the proportional term (P) — the **present** error;
- the integral term (I) — the **past** errors;
- the derivative term (D) — anticipated **future** errors.

- PID control appears in both **simple and complex** systems: as stand-alone controllers, as elements of hierarchical, distributed control systems, etc.

# PID controller

Input/output relation

$$u = k_{\mathrm{p}} e + k_{\mathrm{i}} \int_0^t e(\tau) d\tau + k_{\mathrm{d}} \frac{de}{dt} = k_p \left( e + \frac{1}{T_i} \int_0^t e(\tau) d\tau + T_d \frac{de}{dt} \right).$$

- **Time constant** $T_{\mathrm{i}} = k_{\mathrm{p}}/k_{\mathrm{i}}$ (Integral time); $T_{\mathrm{d}} = k_{\mathrm{d}}/k_{\mathrm{p}}$ (Derivative time)
- Also known as **three-term controllers**.

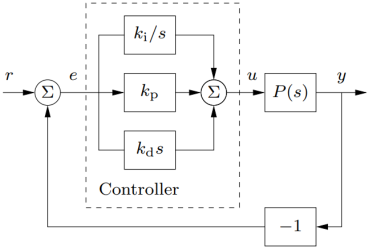



Figure: PID using error feedback

## Example

- Consider a system with dynamics

$$P(s) = \frac{1}{(s+1)^3}.$$

- Consider a controller $C(s)$
- The transfer function from reference to error is

$$G_{\mathrm{er}}(s) = \frac{1}{1 + C(s)P(s)}.$$

# Numerical experiments

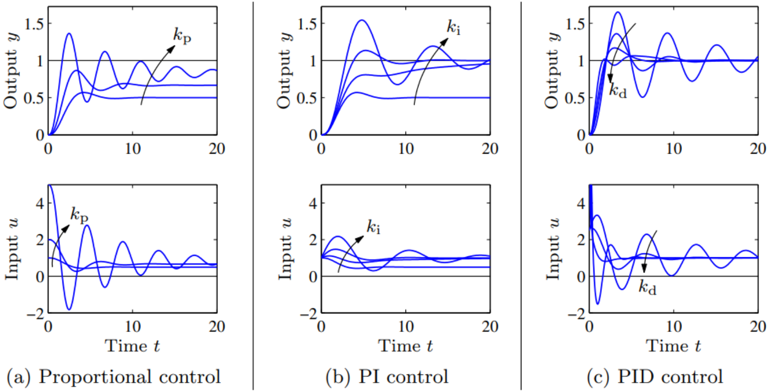



**Figure 11.2:** Responses to step changes in the reference value for a system with a proportional controller (a), PI controller (b), and PID controller (c). The process has the transfer function $P(s) = 1/(s+1)^3$, the proportional controller has parameters $k_p = 1$, 2, and 5, the PI controller has parameters $k_p = 1$, $k_i = 0$, 0.2, 0.5, and 1, and the PID controller has parameters $k_p = 2.5$, $k_i = 1.5$, and $k_d = 0$, 1, 2, and 4.

# Intuition about PID control – P term

The transfer function from reference to error is

$$G_{\mathrm{er}}(s) = \frac{1}{1 + C(s)P(s)} = \frac{1}{1 + k_{\mathrm{p}}P(s)}.$$

- Assuming the closed loop is stable, the steady-state error for a unit step is

$$G_{\mathrm{er}}(0) = \frac{1}{1 + k_{\mathrm{p}}P(0)}.$$

- The error decreases with increasing gain, but the system also become more oscillatory.
- To avoid having a steady-state error, the P term can be changed to

$$u(t) = k_{\mathrm{p}}e(t) + u_{\mathrm{ff}}.$$

  where $u_{\mathrm{ff}}$ is a **feedforward term** (also known as **reset value** — *manually adjusted in early controllers*).
- If the reference value $r$ is constant, we can choose

$$u_{\mathrm{ff}} = \frac{r}{P(0)} = k_{\mathrm{f}}r.$$

  The **zero frequency gain** $P(0)$ might be unknown.

# Intuition about PID control – I term

**Integral action** guarantees that the process output **agrees with the reference in steady state** and provides an alternative to the feedforward term.

- Since this result is **SO IMPORTANT**, we provide a general proof below.

$$u(t) = k_{\mathrm{p}} e(t) + k_{\mathrm{i}} \int_0^t e(\tau) d\tau.$$

- Assume that $u(t)$ and $e(t)$ converge to $u = u_0$ and $e = e_0$

$$u_0 = k_{\mathrm{p}} e_0 + k_{\mathrm{i}} \lim_{t \to \infty} \int_0^t e(\tau) d\tau.$$

- The limit of the right hand side is not finite unless $e(t)$ goes to zero.

Integral control: *if a steady state exists, the error will always be zero*.

- This property is sometimes called the **magic of integral action**.
- Notice that we have **NOT** assumed that the process is linear or time-invariant (we have assumed that there is an equilibrium point).

# Intuition about PID control – I term

The effect of integral action can also be understood from frequency domain analysis.

- The transfer function of a PID controller is

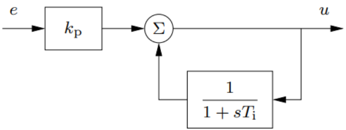
$$C(s) = k_{\mathrm{p}} + \frac{k_{\mathrm{i}}}{s} + k_{\mathrm{d}} s.$$

- This controller has **infinite gain** at zero frequency — **no steady-state error**

$$C(0) = \infty \quad \Rightarrow \quad G_{\mathrm{er}}(0) = \frac{1}{1 + C(0)P(0)} = 0.$$

- Integral action as **Automatic reset** — one of the early inventions (magic of integral action)

$$G_{\mathrm{ue}} = k_{\mathrm{p}} \frac{1 + sT_{\mathrm{i}}}{sT_{\mathrm{i}}} = k_{\mathrm{p}} + \frac{k_{\mathrm{p}}}{sT_{\mathrm{i}}}$$



(a) Integral action (automatic reset)

- Converges more quickly for larger integral gains, but the system also becomes more oscillatory

# Intuition about PID control – D term

The original motivation for derivative feedback was to provide **predictive or anticipatory action**.

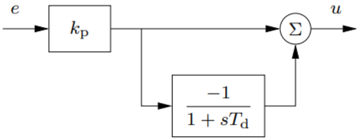
- The combination of the P and D terms can be written

$$u(t) = k_\mathrm{p} e(t) + k_\mathrm{d}\frac{de}{dt} = k_\mathrm{p}\left(e(t) + T_\mathrm{d}\frac{de}{dt}\right) := k_\mathrm{p} e_\mathrm{p},$$

  where $e_\mathrm{p}$ — prediction of the error at time $t + T_\mathrm{d}$ by *linear extrapolation*.

- **Filtered derivative**: difference between the signal and its low-pass filtered version

$$G_\mathrm{ue} = k_\mathrm{p}\left(1 - \frac{1}{1+sT_\mathrm{d}}\right) = k_\mathrm{p}\frac{sT_\mathrm{d}}{1+sT_\mathrm{d}} = \frac{k_\mathrm{d}s}{1+sT_\mathrm{d}}.$$



(b) Derivative action

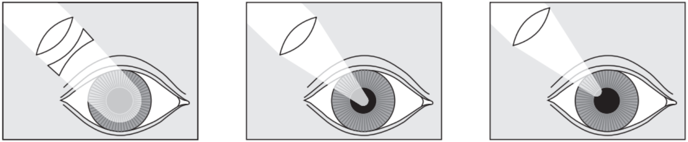
- The transfer function $G_\mathrm{ue}$ acts like a **differentiator** for signals with low frequencies and as a **constant gain** $k_\mathrm{p}$ for high-frequency signals

# PID control in engineering and biological systems

Although PID control was developed in the context of engineering applications, it also appears in nature.

- In biological systems proportional, integral, and derivative action are generated by combining subsystems with dynamical behavior.

**Disturbance attenuation** by feedback in biological systems is often called *adaptation*.



(a) Closed loop (b) Open loop (c) High gain

**Figure 9.21:** Light stimulation of the eye. In (a) the light beam is so large that it always covers the whole pupil, giving closed loop dynamics. In (b) the light is focused into a beam which is so narrow that it is not influenced by the pupil opening, giving open loop dynamics. In (c) the light beam is focused on the edge of the pupil opening, which has the effect of increasing the gain of the system since small changes in the pupil opening have a large effect on the amount of light entering the eye. From Stark [Sta68].

# Outline

# Model reduction (simplification)

*All models are wrong and some are useful*!

- Practical systems are always complex.
- Simplify the models to capture the **essential** properties that are needed for PID design.

- Low-order models can be obtained from first principles.
- Example:
  - Any stable system can be modeled by a static system if its inputs are sufficiently slow.
  - A **first-order** model is sufficient if the storage of mass, momentum, or energy can be captured by only **one variable**
  - System dynamics are of **second order** if the storage of mass, energy, and momentum can be captured by **two state variables**
- A wide range of techniques for model reduction are also available.

# PI for first-order systems

Consider a first-order system with the transfer function $P(s) = \dfrac{b}{s+a}$.

- Consider a PI controller

$$C(s) = k_\mathrm{p} + k_\mathrm{i}\frac{1}{s}.$$

- The closed-loop transfer function from $r$ to $y$ is

$$G_\mathrm{yr}(s) = \frac{PC}{1+PC} = \frac{bk_\mathrm{p}s + k_\mathrm{i}}{s^2 + (a + bk_\mathrm{p})s + k_\mathrm{i}}.$$

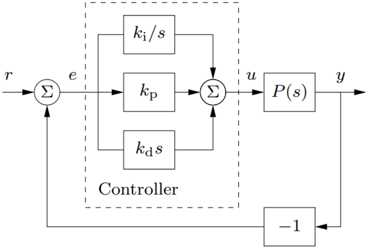



Figure: PID using error feedback

- Requiring that the closed loop system have the characteristic polynomial

$$p(s) = s^2 + a_1 s + a_2.$$

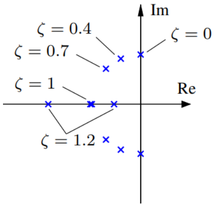
- Controller parameters are

$$k_\mathrm{p} = \frac{a_1 - a}{b}, \qquad k_\mathrm{i} = \frac{a_2}{b}.$$

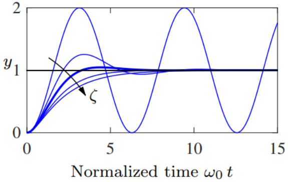
# PI for first-order systems

**Table 7.1:** Properties of the step response for a second-order system $\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = k\omega_0^2 u$ with $0 < \zeta \leq 1$.

| Property | Value | $\zeta = 0.5$ | $\zeta = 1/\sqrt{2}$ | $\zeta = 1$ |
|---|---|---|---|---|
| Steady-state value | $k$ | $k$ | $k$ | $k$ |
| Rise time (inverse slope) | $T_r = e^{\varphi/\tan\varphi}/\omega_0$ | $1.8/\omega_0$ | $2.2/\omega_0$ | $2.7/\omega_0$ |
| Overshoot | $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$ | 16% | 4% | 0% |
| Settling time (2%) | $T_s \approx 4/\zeta\omega_0$ | $8.0/\omega_0$ | $5.6/\omega_0$ | $4.0/\omega_0$ |




(a) Eigenvalues




(b) Step responses

Figure: Step response for a second-order system.

# PID control for Second-order Systems

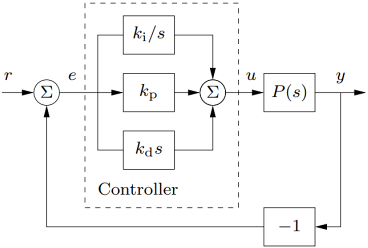



Figure: PID using error feedback

- Consider a second-order plant:

$$P(s) = \frac{b_0}{s^2 + a_1 s + a_0}$$

- Consider a PID controller

$$C(s) = k_p + k_i \frac{1}{s} + +k_d s.$$

- The closed-loop transfer function from $r$ to $y$ is

$$G_{yr}(s) = \frac{PC}{1 + PC}$$

How should the controller $C(s)$ be designed to ensure that the closed-loop system is **stable** and its **step response has zero steady-state error**?

# Case 1: Proportional (P) Control

A **proportional (P) controller** uses a constant gain $k_\mathrm{p}$:

$$C(s) = k_\mathrm{p} \qquad \Leftrightarrow \qquad u(t) = k_\mathrm{p} e(t)$$

- Closed-loop transfer function:

$$G_\mathrm{yr}(s) = \frac{PC}{1 + PC} = \frac{k_\mathrm{p} b_0}{s^2 + a_1 s + (a_0 + k_\mathrm{p} b_0)}$$

- P control can accelerate the response of a second-order system by changing the natural frequency $\omega_0^2 = (a_0 + k_\mathrm{p} b_0)$
- To ensure stability, we need $a_1 > 0$ and $a_0 + k_\mathrm{p} b_0 > 0$.
- P control can stabilize only some systems because it adjusts one coefficient of the characteristic equation.

For $a_0 \neq 0$, the closed-loop system step response will have a **constant finite steady-state error**.

$$G_\mathrm{yr}(0) = \frac{k_\mathrm{p} b_0}{a_0 + k_\mathrm{p} b_0} < 1.$$

# Case 2: Proportional-Integral (PI) Control

A **proportional-integral (PI) controller** uses a proportional gain $K_p$ and an integral gain $K_i$:

$$C(s) = k_{\mathrm{p}} + \frac{k_{\mathrm{i}}}{s} \qquad \Leftrightarrow \qquad u(t) = k_{\mathrm{p}} e(t) + k_{\mathrm{i}} \int_0^t e(\tau) d\tau$$

- Closed-loop transfer function:

$$G_{\mathrm{yr}}(s) = \frac{PC}{1+PC} = \frac{b_0(k_{\mathrm{p}} s + k_{\mathrm{i}})}{s^3 + a_1 s^2 + (a_0 + k_{\mathrm{p}} b_0)s + k_{\mathrm{i}} b_0}$$

- **Zero steady-state error** if the closed-loop system is stable

$$G_{\mathrm{yr}}(0) = \frac{b_0 k_{\mathrm{i}}}{k_{\mathrm{i}} b_0} = 1.$$

We achieved the steady-state error specification but the closed-loop system might still be unstable if $a_1 < 0$.

# Case 3: PID Control

A **proportional-integral-derivative (PID) controller** uses a proportional gain $K_p$, an integral gain $K_i$, and a derivative gain $K_d$:

$$C(s) = k_\mathrm{p} + \frac{k_\mathrm{i}}{s} + k_\mathrm{d} s \qquad u = k_\mathrm{p} e + k_\mathrm{i} \int_0^t e(\tau) d\tau + k_\mathrm{d} \frac{de}{dt}$$

- Closed-loop transfer function:

$$G_\mathrm{yr}(s) = \frac{PC}{1+PC} = \frac{b_0(k_\mathrm{p} s + k_\mathrm{i} + k_\mathrm{d} s^2)}{s^3 + (a_1 + k_\mathrm{d} b_0)s^2 + (a_0 + k_\mathrm{p} b_0)s + k_\mathrm{i} b_0}$$

- The coefficients of the characteristic polynomial can be set **arbitrarily** via an appropriate choice of $k_\mathrm{p}$, $k_\mathrm{i}$, $k_\mathrm{d}$

For a second-order plant, PID control can guarantee

- *stability*, *good transient behavior*, and *zero steady-state step error*.

# PID Control Example

## Example

Consider the plant

$$P(s) = \frac{1}{s^2 - 3s - 1}$$

Design a PID controller $C(s)$ to achieve step response with zero steady-state error and place the closed-loop system poles at $-1$, $-2$, $-3$

- PID controller: $C(s) = k_\mathrm{p} + \frac{k_\mathrm{i}}{s} + k_\mathrm{d} s$
- Closed-loop transfer function:

$$G_\mathrm{yr}(s) = \frac{PC}{1+PC} = \frac{k_\mathrm{d} s^2 + k_\mathrm{p} s + k_\mathrm{i}}{s^3 + (k_\mathrm{d} - 3)s^2 + (k_\mathrm{p} - 1)s + k_\mathrm{i}}$$

- Matching coefficients with

$$\begin{aligned} p(s) &= (s+1)(s+2)(s+3) \\ &= (s^2 + 3s + 2)(s+3) \\ &= s^3 + 6s^2 + 11s + 6, \end{aligned}$$

we have $k_\mathrm{d} = 9$, $k_\mathrm{p} = 12$, $k_\mathrm{i} = 6$.

# Outline

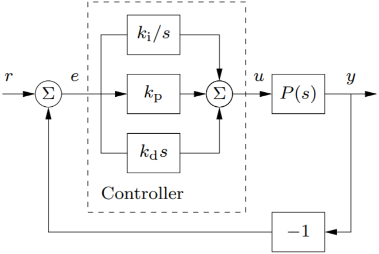



Figure: PID using error feedback

PID control

- the proportional term (P) — the **present** error;
- the integral term (I) — the **past** errors;
- the derivative term (D) — anticipated **future** errors.

- **Magic of integral action**

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau) d\tau.$$

$$\Rightarrow \quad u_0 = k_p e_0 + k_i \lim_{t \to \infty} \int_0^t e(\tau) d\tau.$$

- PID controller for lower-order (1st and 2nd order) systems