ECE 171A: Linear Control System Theory
Lecture 21: Review

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Outline

Review: L11 - L20

Examples: Nyquist plot

Midterm II

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Review: L11 - L20

Examples: Nyquist plot

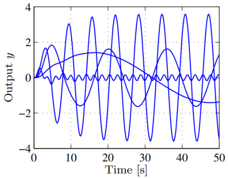
Midterm II

L11 - Input/output responses (II)

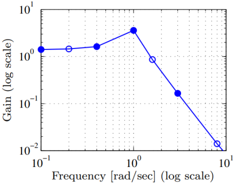
- **Impulse response**

$$h(t) = \int_0^t Ce^{A(t-\tau)}B\delta(\tau)d\tau + D\delta(t) = Ce^{At}B + D\delta(t).$$

- **Frequency responses**



(a) Time domain simulations



(b) Frequency response

- **The convolution equation**

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t).$$

another version is $y(t) = \underbrace{Ce^{At}x(0)}_{\text{Initial response}} + \underbrace{\int_0^t h(t-\tau)u(\tau)d\tau}_{\text{forced response}}$

L12: Transfer function (I)

- Transient response and steady-state response

$$y(t) = \underbrace{Ce^{At}\left(x(0) - (sI - A)^{-1}B\right)}_{\text{transient}} + \underbrace{\left(C(sI - A)^{-1}B + D\right)e^{st}}_{\text{steady-state}}$$

- Transfer function

$$G(s) = C(sI - A)^{-1}B + D.$$

  - Express the steady-state solution of a stable linear system forced by a sinusoidal input

$$u(t) = \sin(\omega t) \to y_{\text{ss}} = |G(i\omega)| \sin(\omega t + \angle G(i\omega))$$

- **Frequency domain modeling**: Modeling a system through its response to sinusoidal and exponential signals.
  - We represent the dynamics of the system in terms of the generalized frequency s rather than the time domain variable t.
  - The **transfer function** provides a complete representation of a linear system in the frequency domain.

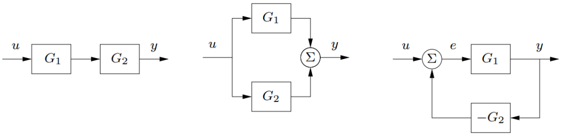
L13: Transfer function (II)

- Transfer function for linear ODEs

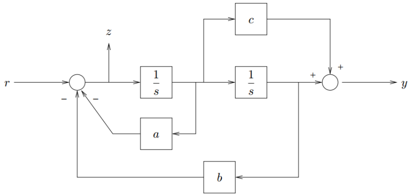
$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \ldots + b_0 u,$$

$$G(s) = \frac{b_m s^m + b_{m-1}s^{n-1} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}.$$

- Block diagram with transfer functions



(a) $G_{yu} = G_2 G_1$ (b) $G_{yu} = G_1 + G_2$ (c) $G_{yu} = \frac{G_1}{1 + G_1 G_2}$



Common transfer functions

Type	System	Transfer function
Integrator	$\dot{y} = u$	$\frac{1}{s}$
Differentiator	$y = \dot{u}$	s
First-order system	$\dot{y} + ay = u$	$\frac{1}{s+a}$
Double integrator	$\ddot{y} = u$	$\frac{1}{s^2}$
Damped oscillator	$\ddot{y} + 2\zeta\omega_0\dot{y} + \omega_0^2 y = u$	$\frac{1}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$
State-space system	$\dot{x} = Ax + Bu$ $y = Cx + Du$	$C(sI - A)^{-1}B + D$
PID controller	$y = k_\mathrm{p} u + k_\mathrm{d}\dot{u} + k_\mathrm{i}\int u$	$k_\mathrm{p} + k_\mathrm{d} s + \frac{k_\mathrm{i}}{s}$
Time delay	$y(t) = u(t - \tau)$	$e^{-\tau s}$

Table: Transfer functions for some common linear time-invariant systems.

L14: Zeros, Poles and Bode plot

- The **features** of a transfer function are often associated with **important system properties**.
  - zero frequency gain
  - the locations of the poles and zeros: Poles — modes of a system; Zeros – Block transmission of certain signals

Poles (eigenvalues) of the matrix A = Poles of the transfer function $G(s)$

- The frequency response $G(i\omega)$ can be represented by two curves — **Bode plot**
  - **Gain curve**: gives $|G(i\omega)|$ as a function of frequency ω — log/log scale (traditionally in dB — $20 \log |G(i\omega)|$; we often consider $\log |G(i\omega)|$)
  - **Phase curve**: gives $\angle G(i\omega)$ as a function of frequency ω — log/linear scale in degrees

L15: Bode plot and Routh-Hurwitz stability

- The **Bode plot** gives a quick overview of a **stable** linear system.

$$u(t) = \sin(\omega t) \qquad \rightarrow \qquad y_{\text{ss}} = |G(i\omega)| \sin(\omega t + \angle G(i\omega))$$

Theorem

Consider a Routh table from the polynomial $a(s)$ in

$$G(s) = \frac{b(s)}{a(s)}.$$

- *The number of sign changes in the first column of the Routh table is equal to the number of roots of $a(s)$ in the closed right half-plane.*

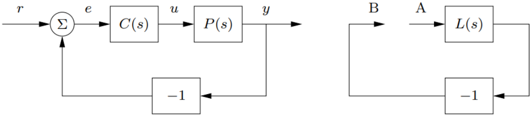
Corollary (BIBO Stability of LTI Systems)

The system $G(s)$ is **BIBO stable** *if and only if there are no sign changes in the first column of its Routh table.*

L16: Loop transfer functions and Nyquist plot

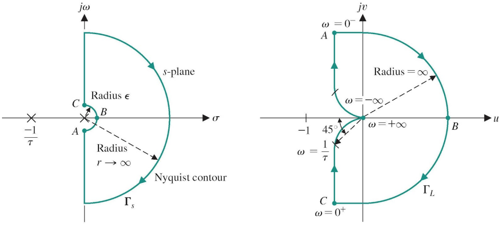
- Nyquist's idea was to first investigate conditions under which oscillations can occur in a feedback loop – The **Loop transfer function**:

$$L(s) = P(s)C(s).$$



(a) Closed loop system (b) Open loop system

- **Nyquist plot** and Simplified **Nyquist criterion**



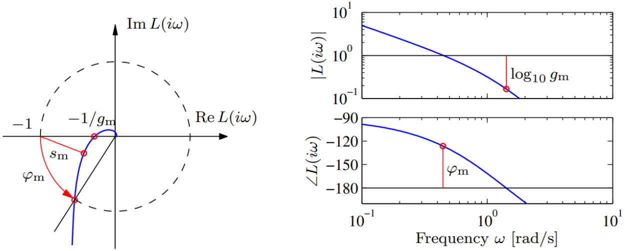
(a) (b)

L17: Nyquist Criterion and Stability margins

Theorem (Nyquist's Stability Criterion)

Consider a unity feedback control system with open-loop transfer function $L(s)$. Let Γ be a Nyquist contour. The system is stable if and only if the number of counterclockwise encirclements of $-1 + i0$ by the Nyquist plot $L(\Gamma)$ is equal to the number of poles of $L(s)$ inside Γ.

Classical robustness measures: stability margin, phase margin, gain margin



(a) Nyquist plot (b) Bode plot

L18: Bode's relations and Root locus

- **Minimum phase systems**: they have the smallest phase lag of all systems with the same gain curve — for these systems
  - No time delays or poles and zeros in the right half-plane.
  - Have the property that $\log|P(s)|/s \to 0$ as $s \to \infty$ for $\mathrm{Re}(s) \geq 0$.
- For minimum phase systems, the phase is uniquely given by the shape of the gain curve and vice versa
- **Root locus**: a graph of the roots of $a_{\mathrm{cl}}(s)$ as the gain k is varied from 0 to ∞.
  - The plot of root locus will have n branches.
  - Each branch starts at a different open-loop pole.
  - m of the branches end at different open-loop zeros.
  - The remaining $n - m$ branches go to infinity.

L19: PID control (I)

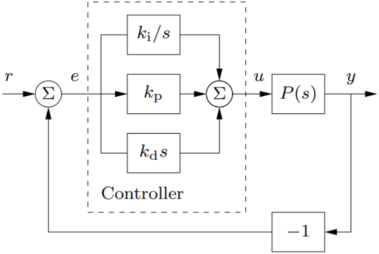


Figure: PID using error feedback

PID control

- the proportional term (P) — the **present** error;
- the integral term (I) — the **past** errors;
- the derivative term (D) — anticipated **future** errors.

- **Magic of integral action**

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau) d\tau.$$

$$\Rightarrow \quad u_0 = k_p e_0 + k_i \lim_{t\to\infty} \int_0^t e(\tau) d\tau.$$

- PID controller for lower-order (1st and 2nd order) systems

PID Control Example

Example

Consider the plant

$$P(s) = \frac{1}{s^2 - 3s - 1}$$

Design a PID controller $C(s)$ to achieve step response with zero steady-state error and place the closed-loop system poles at -1, -2, -3

- PID controller: $C(s) = k_{\mathrm{p}} + \frac{k_{\mathrm{i}}}{s} + k_{\mathrm{d}} s$
- Closed-loop transfer function:

$$G_{\mathrm{yr}}(s) = \frac{PC}{1+PC} = \frac{k_{\mathrm{d}} s^2 + k_{\mathrm{p}} s + k_{\mathrm{i}}}{s^3 + (k_{\mathrm{d}} - 3)s^2 + (k_{\mathrm{p}} - 1)s + k_{\mathrm{i}}}$$

- Matching coefficients with

$$\begin{aligned} p(s) &= (s+1)(s+2)(s+3) \\ &= (s^2 + 3s + 2)(s+3) \\ &= s^3 + 6s^2 + 11s + 6, \end{aligned}$$

  we have $k_{\mathrm{d}} = 9$, $k_{\mathrm{p}} = 12$, $k_{\mathrm{i}} = 6$.

L20: PID control (II)

- **Ziegler-Nichols' Tuning**
- **Tuning based on the FOTD model**
- Relay Feedback (Automatic tuning; not required in this course)

- Many aspects of a control system can be understood from linear models.
- However, some nonlinear phenomena must be taken into account
- Windup can occur in any controller with integral action.
- There are many methods to avoid windup.

Outline

Review: L11 - L20

Examples: Nyquist plot

Midterm II

Example 4

$$L(s) = \frac{1}{s+1}$$

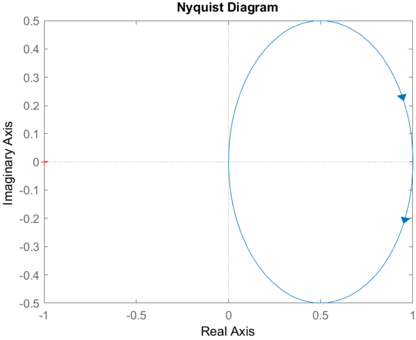


Figure: Nyquist plot for $L(s) = \frac{1}{s+1}$

$$Z = N + P = 0$$

Then,

$$G_{\text{yr}} = \frac{L(s)}{1 + L(s)} = \frac{1}{s+2}$$

Example 5

$$L(s) = \frac{1}{(s+1)^2}$$

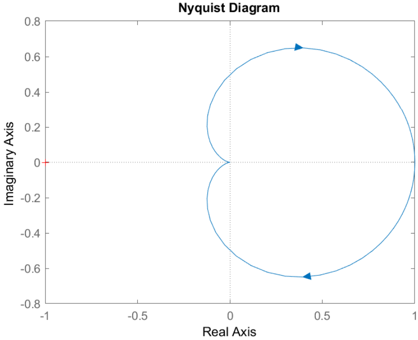


Figure: Nyquist plot for $L(s) = \frac{1}{(s+1)^2}$

$$Z = N + P = 0$$

Then,

$$G_{\mathrm{yr}} = \frac{L(s)}{1 + L(s)} = \frac{1}{s^2 + 2s + 2}$$

Closed-loop poles

$$p_{1,2} = -1 \pm 1i$$

Example 6

$$L(s) = \frac{1}{s(s+1)}$$

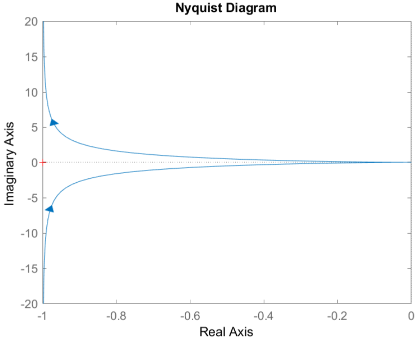


Figure: Nyquist plot for $L(s) = \frac{1}{s(s+1)}$

$$Z = N + P = 0$$

Then,

$$G_{\text{yr}} = \frac{L(s)}{1 + L(s)} = \frac{1}{s^2 + s + 1}$$

Closed-loop poles

$$p_{1,2} = -0.5 \pm 0.866i$$

Example 7

$$L(s) = \frac{1}{s(s+1)(s+0.5)}$$

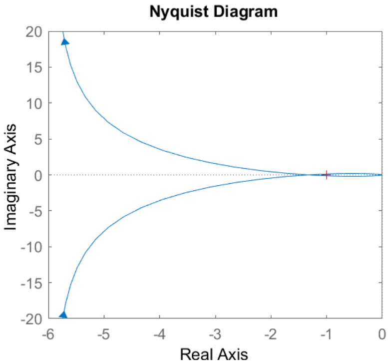


Figure: Nyquist plot for $L(s) = \frac{1}{s(s+1)(s+0.5)}$

$$Z = N + P = 2$$

Then,

$$G_{\mathrm{yr}} = \frac{L(s)}{1+L(s)}$$
$$= \frac{1}{s^3 + 1.5s^2 + 0.5s + 1}$$

Closed-loop poles

$$p_{1,2} = 0.0416 \pm 0.7937i$$
$$p_3 = -1.5832$$

Outline

Midterm II

- Midterm exam (II) — in class, May 18 (Wednesday in Week 8)
  - **Scope**: Lectures 11 - 21, HW4 - HW6, HW7 (Q1, Q2), DI 5-8; (Reading materials in the textbook)
  - Closed book, closed notes, closed external links.
  - **Come on time** (1 or 2 minutes early if you can; we will start at 9:00 am promptly)
  - No MATLAB is required. No graphing calculators are permitted. A basic arithmetic calculator is allowed.
  - The exams must be done in a blue book. Bring a blue book with you.
  - **No collaboration and discussions are allowed**. It is dishonest to cheat on exams. Instances of academic dishonesty will be referred to the Office of Student Conduct for adjudication. *You don't want to take a risk for such a small thing*.

Midterm II

Three problems

- Problem 1: True or False
- Problem 2: Bode and Nyquist
- Problem 3: Feedback Control

Good Luck!