ECE 171A: Linear Control System Theory
Lecture 22: Performance specifications

Yang Zheng

Assistant Professor, ECE, UCSD

May 20, 2022

Reading materials: Ch 12.1, 12.2

Outline

Midterm II and Survey

Midterm II: Many of you did well

- Maximum: 98
- Above 90: 9; 80 - 90: 11
- Mean: 70.5; Median: 71.
- STD DEV: 14.8

Midterm I:

- Maximum: 102
- Above 90: 10; 80 - 90: 10
- Mean: 67.3; Median: 69
- STD DEV: 19.4

Come to office hours or send us an email if you want to chat.

- **Grading**: Take a maximum of the following methods
  1. HW 40% + Midterm I 10 % + Midterm II 10 % + Final 40 %
  2. HW 40% + Midterm I 5 % + Midterm II 10 % + Final 45 %
  3. HW 40% + Midterm I 10 % + Midterm II 5 % + Final 45 %
- **Survey** on midterms and final; Please spend 2 minutes filling it out by next Monday.

`https://forms.gle/TkPt3H6McBMZHdFe8`

Schedule

- The rest of this course:

	May 18	Midterm II - in class		
	May 20	L22: Performance specification	Ch 12.1, 12.2	
9	May 23	L23: Loop shaping	Ch 12.3	
	May 23	D9: Review on midterms		
	May 25	L24: Uncertainty and robustness	Ch 13.1, 13.2	Homework 8
	May 27	L25: Fundamental limits (I)	Ch 14.1, 14.2	
10	May 30	Memorial Day observance (no lecture)		
	May 30	Memorial Day observance (no disucssion)		
	Jun 01	L26: Fundamental limits (II)	Ch 14.5	
	Jun 03	L27: Review		
11	Jun 08	Final Exam		

Hope you will continue to have fun in the final two weeks.
Let's keep in touch even after the course.

Outline

Feedback control systems

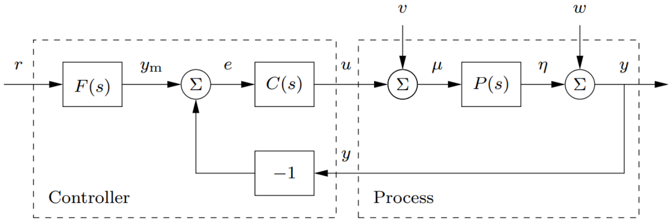


Figure: Block diagram of a control system with two degrees of freedom.

- The 2 DOF controller:
  - a feedback block $C(s)$
  - a feedforward block $F(s)$.
- The external signals
  - the reference signal r,
  - the load disturbance v,
  - the measurement noise w.
- The measured output is y, and the control signal is u.

Loop shaping: design the behavior of the *closed-loop system* by focusing on the *open-loop transfer function*.

- **Example — the Nyquist criterion**: we used the Nyquist plot for the open loop transfer function to determine the stability of the closed loop system

Closed-loop transfer functions

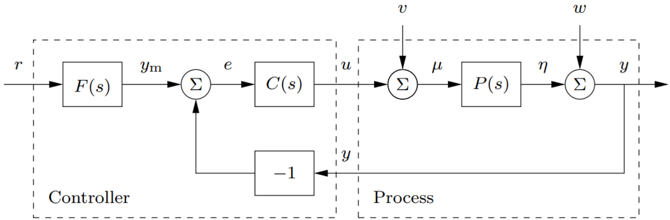


Figure: Block diagram of a control system with two degrees of freedom.

- The feedback control system is composed of linear elements;
- The relations between the signals can be expressed via transfer functions.

The overall system has three external inputs: the **reference** r, the **load disturbance** v, and the **measurement noise** w.

- Any of the remaining signals can be relevant for design, but the most common ones are
  - the error e, the input u, and the output y.

Closed-loop transfer functions

Table 12.1: Transfer functions relating the signals of the control system in Figure 12.1. The external inputs are the reference signal r, load disturbance v, and measurement noise w, represented by each row. The columns represent the measured signal y, control input u, error e, process input μ, and process output η that are most relevant for system performance.

y	u	e	μ	η	
$\frac{PCF}{1+PC}$	$\frac{CF}{1+PC}$	$\frac{F}{1+PC}$	$\frac{CF}{1+PC}$	$\frac{PCF}{1+PC}$	r
$\frac{P}{1+PC}$	$\frac{-PC}{1+PC}$	$\frac{-P}{1+PC}$	$\frac{1}{1+PC}$	$\frac{P}{1+PC}$	v
$\frac{1}{1+PC}$	$\frac{-C}{1+PC}$	$\frac{-1}{1+PC}$	$\frac{-C}{1+PC}$	$\frac{-PC}{1+PC}$	w

For most control designs, we focus on the following subset — the **Gang of Six**

$$G_{\mathrm{yr}} = \frac{PCF}{1+PC}, \quad -G_{\mathrm{uv}} = \frac{PC}{1+PC}, \quad G_{\mathrm{yv}} = \frac{P}{1+PC}$$

$$G_{\mathrm{ur}} = \frac{CF}{1+PC}, \quad -G_{\mathrm{uw}} = \frac{C}{1+PC}, \quad G_{\mathrm{yw}} = \frac{1}{1+PC}$$

Sensitivity functions: Gang of four

The response of the system to load disturbances v and measurement noise w is of particular importance — referred to as **sensitivity functions**.

- Sensitivity function

$$S = \frac{1}{1+PC} \qquad \rightarrow \qquad G_{\text{yw}}$$

- Complementary sensitivity function

$$T = \frac{PC}{1+PC} \qquad \rightarrow \qquad -G_{\text{uv}}$$

- Load sensitivity function

$$PS = \frac{P}{1+PC} \qquad \rightarrow \qquad G_{\text{yv}}$$

- Noise sensitivity function

$$CS = \frac{C}{1+PC} \qquad \rightarrow \qquad -G_{\text{uw}}$$

Important in feedback control design — called the **Gang of Four**

- The remaining two in the **Gang of Six** are

$$G_{\text{yr}} = \frac{PCF}{1+PC}, \qquad G_{\text{ur}} = \frac{CF}{1+PC}$$

- feedback controller $C(s) \rightarrow$ provide good response w.r.t. w and v;
- feedforward controller $F(s)$ $\rightarrow$ obtain good reference tracking w.r.t. r.

Sensitivity functions: Gang of four

Gang of Four

$$S = \frac{1}{1+PC} \qquad PS = \frac{P}{1+PC}$$
$$T = \frac{PC}{1+PC} \qquad CS = \frac{C}{1+PC}$$

Many interesting properties (assuming stability for the reference tracking)

- It directly follows that

$$S(s) + T(s) = 1, \qquad \forall s \in \mathbb{C}$$

- The loop transfer function PC will typically go to zero for large s, which implies that $T(s) \to 0$ and $S(s) \to 1$ as $s \to \infty$.
- It will not be possible to track very high-frequency reference signals $|G_{\mathrm{yr}} = PT| \to 0$
- For controllers with integral action, PC will typically go to infinity for small s — **low frequency references are tracked well**.

Outline

Sensitivity Functions

Performance specifications

Summary

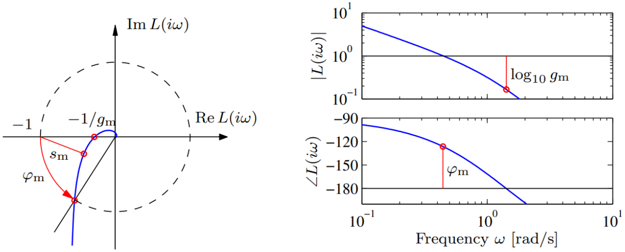
Performance specifications

Specifications capture **robustness** to process variations and **performance** w.r.t.

- the ability to **track** reference signals and **attenuate** load disturbances **without injecting** too much measurement noise.
- Expressed in terms of transfer functions (the Gang of Six) and the loop transfer function, using features of their **time and frequency responses**.

Robustness to process variations

- *Loop transfer function*: Gain/phase margin g_m, φ_m, and stability margin s_m.



(a) Nyquist plot (b) Bode plot

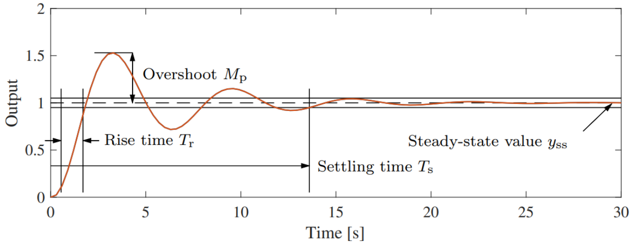
Performance specifications

Specifications capture **robustness** to process variations and **performance** w.r.t.

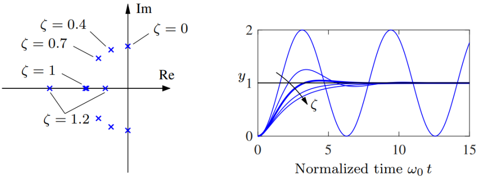
- the ability to **track** reference signals and **attenuate** load disturbances **without injecting** too much measurement noise.
- Expressed in terms of transfer functions (the Gang of Six) and the loop transfer function, using features of their **time and frequency responses**.

Features of step responses

- Overshoot, rise time, and settling time



Connections between time and frequency domains



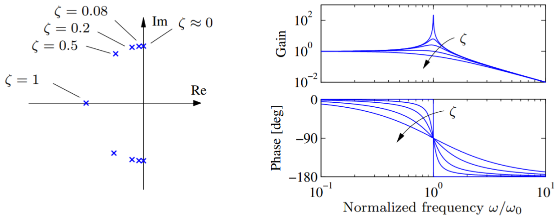
(a) Eigenvalues (b) Step responses

Table 7.1: Properties of the step response for a second-order system $\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = k\omega_0^2 u$ with $0 < \zeta \leq 1$.

Property	Value	$\zeta = 0.5$	$\zeta = 1/\sqrt{2}$	$\zeta = 1$
Steady-state value	k	k	k	k
Rise time (inverse slope)	$T_r = e^{\varphi/\tan\varphi}/\omega_0$	$1.8/\omega_0$	$2.2/\omega_0$	$2.7/\omega_0$
Overshoot	$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$	16%	4%	0%
Settling time (2%)	$T_s \approx 4/\zeta\omega_0$	$8.0/\omega_0$	$5.6/\omega_0$	$4.0/\omega_0$

Roughly speaking, the behavior of **time responses** for short times is related to the behavior of **frequency responses** at high frequencies, and vice versa.

Connections between time and frequency domains



(a) Eigenvalues (b) Frequency responses

Table 7.2: Properties of the frequency response for a second-order system $\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = k\omega_0^2 u$ with $0 < \zeta \le 1$.

Property	Value	$\zeta = 0.1$	$\zeta = 0.5$	$\zeta = 1/\sqrt{2}$
Zero frequency gain	M_0	k	k	k
Bandwidth	$\omega_b = \omega_0\sqrt{1 - 2\zeta^2 + \sqrt{(1-2\zeta^2)^2 + 1}}$	$1.54\,\omega_0$	$1.27\,\omega_0$	ω_0
Resonant peak gain	$M_r = \begin{cases} k/(2\zeta\sqrt{1-\zeta^2}) & \zeta \le \sqrt{2}/2, \\ \text{N/A} & \zeta > \sqrt{2}/2 \end{cases}$	$5\,k$	$1.15\,k$	k
Resonant frequency	$\omega_{mr} = \begin{cases} \omega_0\sqrt{1-2\zeta^2} & \zeta \le \sqrt{2}/2, \\ 0 & \zeta > \sqrt{2}/2 \end{cases}$	ω_0	$0.707\omega_0$	0

Response to Reference Signals

The responses of the output y and the control signal u to the reference r are

$$G_{\mathrm{yr}} = \frac{PCF}{1+PC}, \qquad G_{\mathrm{ur}} = \frac{CF}{1+PC}.$$

- Peak (or resonant) value M_{r}, the peak frequency ω_{mr} the bandwidth ω_{r}.

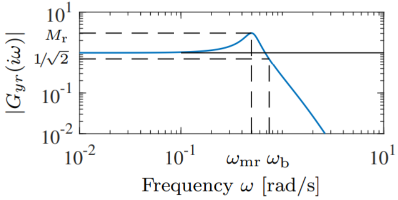


Figure: Gain curve of the transfer function G_{yr}

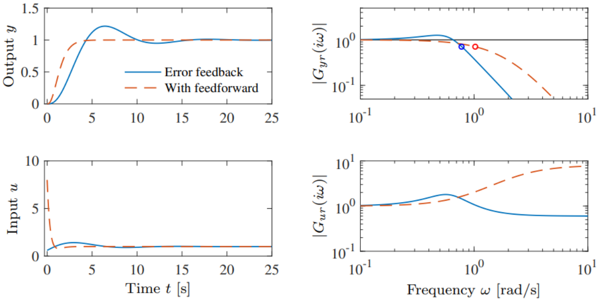
- We can choose $F = 1$
- But in many cases it is useful to retain the ability to shape the input/output response by using $F(s) \neq 1$ —- **the full Gang of Six**

Example

Example

Consider a process with the transfer function and PI controller

$$P(s) = \frac{1}{(s+1)^3}, \qquad C(s) = 0.6 + \frac{0.5}{s}$$



(a) Step responses (b) Frequency responses

With feedforward controller $F(s)$

- **Faster response, no overshoot**
- **Larger bandwidth, no peak**

Response to Load Disturbances and Measurement Noise

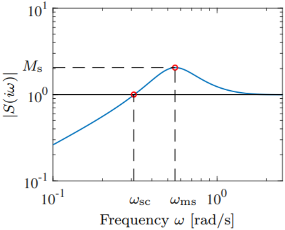
Sensitivity function

$$S = \frac{1}{1+PC} \quad \rightarrow \quad G_{\text{yw}}$$

Load sensitivity function

$$PS = \frac{P}{1+PC} \quad \rightarrow \quad G_{\text{yv}}$$

- The sensitivity function S directly shows how feedback $C(s)$ influences the response of the output to disturbances w
- Disturbances with frequencies such that $|S(i\omega)| < 1$ are **attenuated**; such that $|S(i\omega)| > 1$ are **amplified** by feedback
- Sensitivity **crossover** frequency ω_{sc} is the lowest frequency such that $|S(i\omega)| = 1$.



(a) Gain curves

Response to Load Disturbances and Measurement Noise

Load sensitivity function $PS = \dfrac{P}{1+PC} \quad \rightarrow \quad G_{\mathrm{yv}}$

- Load disturbances typically have low frequencies.
- For strictly proper plant, $P(s)$ is small for low frequencies.
- For processes with $P(0) \neq 0$ and controllers with integral action, we have

$$C(s) \approx \frac{k_{\mathrm{i}}}{s} \qquad \Rightarrow \quad G_{\mathrm{yv}} \approx \frac{1}{C} = \frac{s}{k_{\mathrm{i}}}$$

  for small s (low frequencies)

A controller with integral action thus **attenuates disturbances with low frequencies** effectively, and the gain k_{i} is a measure of disturbance attenuation.

Similar analysis for other transfer functions, e.g., **high-frequency roll-off**

$$-G_{\mathrm{uw}} = \frac{C}{1+PC} \approx C \qquad \text{for large } s.$$

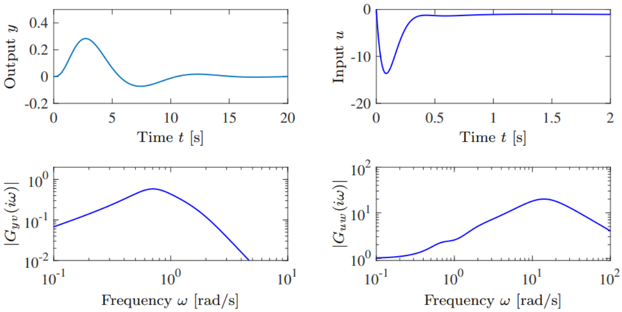
- It is useful to filter the derivative so that $C(s)$ go to zero for large s.

Example

Example

Consider a process with the transfer function and PID controller

$$P(s) = \frac{1}{(s+1)^3}, \qquad C(s) = \frac{k_d s^2 + k_p s + k_i}{s(s^2 T_f^2/2 + sT_f + 1)}$$



(a) Output load response　　(b) Input noise response

Outline

Summary

- **Sensitivity functions**: for most control designs we focus on the following subset — the **Gang of Six**

$$G_{\mathrm{yr}} = \frac{PCF}{1+PC}, \quad -G_{\mathrm{uv}} = \frac{PC}{1+PC}, \quad G_{\mathrm{yv}} = \frac{P}{1+PC}$$
$$G_{\mathrm{ur}} = \frac{CF}{1+PC}, \quad -G_{\mathrm{uw}} = \frac{C}{1+PC}, \quad G_{\mathrm{yw}} = \frac{1}{1+PC}$$

- Specifications capture **robustness** to process variations and **performance** w.r.t.
  - the ability to **track** reference signals and **attenuate** load disturbances **without injecting** too much measurement noise.
- Expressed in terms of transfer functions (e.g., the Gang of Six) and the loop transfer function, using features of their **time and frequency responses**.