ECE 171A: Linear Control System Theory
Lecture 25: Fundamental Limits (I)

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Reading materials: Ch 14.1, 14.2

Final Exam

- Final Exam — 8:00 am - 10:30 am, June 08
  - 4 Problems in total
  - Lectures 1 - 27, HW1 - HW8, DI 1-9; (Reading materials in the textbook)
  - This final exam is closed book but you can bring one sheet of notes (page maximum size: Letter; can be double-sided).
  - No MATLAB is required. No graphing calculators are permitted. You need a basic arithmetic calculator for simple calculations.
  - **Come on time** (1 or 2 minutes early if you can; we will start at 8:00 am promptly)
  - The exams must be done in a blue book. Bring a blue book with you.
  - **No collaboration and discussions are allowed**. It is dishonest to cheat on exams. Instances of academic dishonesty will be referred to the Office of Student Conduct for adjudication. *You don't want to take a risk for such a small thing*.

Course evaluation

- You should have got an email from UCSD Online Evaluations to evaluate ECE 171A
- **Deadline**: Monday, June 6 at 11:59 PM
- Your responses are completely anonymous.
- It's your opportunity to let your voices be heard (by me and the university).
- Please give some **thoughtful**, **truthful**, and **constructive** feedback.
- If you like the course, please say it explicitly and we'd love to hear it 🌹
- If you think some aspects can be improved, we are more than happy to know them 🌹 🌹

Many thanks for your efforts and your time into this course!

Outline

Robust stability (continue from Lecture 24)

Fundamental limits - System design considerations

Summary

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Robust stability (continue from Lecture 24)

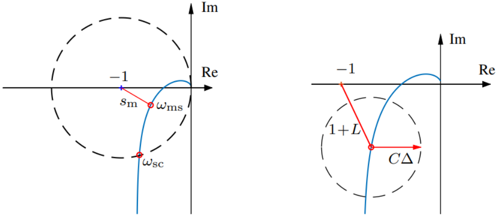
Fundamental limits - System design considerations

Summary

Robust stability

Robust stability: when can we formally show that the stability of a system is robust with respect to process variations?

- **Nyquist criterion**: a powerful and elegant way to study the effects of uncertainty.
- The stability margin s_m is a good robustness measure.



(a) Nyquist plot (b) Additive uncertainty

Figure 13.7: Illustrations of robust stability in Nyquist plots. The plot (a) shows the stability margin $s_m = 1/M_s$. The plot (b) shows the Nyquist curve and the circle shows uncertainty due to stable additive process variations Δ.

Robust stability - explicit conditions

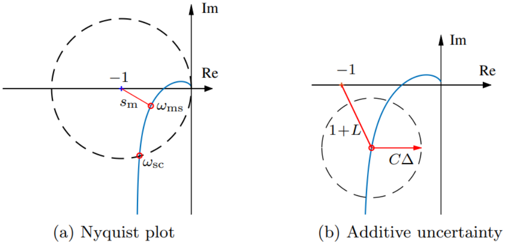


Figure 13.7: Illustrations of robust stability in Nyquist plots. The plot (a) shows the stability margin $s_m = 1/M_s$. The plot (b) shows the Nyquist curve and the circle shows uncertainty due to stable additive process variations Δ.

- If the process is changed from $P(s)$ to $P(s) + \Delta(s)$, the loop transfer function changes from $P(s)C(s)$ to

$$(P(s) + \Delta(s))C(s).$$

- Assume that $\Delta(s)$ is stable, the closed-loop system remains stable as long as the perturbed loop transfer function

$$(P + \Delta)C$$

never reaches the critical point -1.

Robust stability - explicit conditions

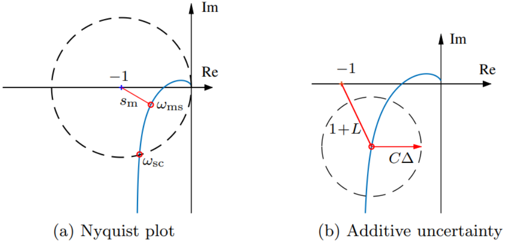


Figure 13.7: Illustrations of robust stability in Nyquist plots. The plot (a) shows the stability margin $s_m = 1/M_s$. The plot (b) shows the Nyquist curve and the circle shows uncertainty due to stable additive process variations Δ.

- The distance from -1 to $L = PC$ is $|1 + L|$.
- The perturbed Nyquist curve will not reach -1 provided that

$$|C\Delta| < |1 + L| \tag{1}$$

- (1) holds if

$$|\Delta| < \left|\frac{1+L}{C}\right|, \quad \text{or} \quad |\delta| < \left|\frac{1+L}{L}\right| = \frac{1}{|T|}, \quad \text{where} \quad \delta = \frac{\Delta}{P} \tag{2}$$

Robust stability - explicit conditions

The condition (2) must be valid all all points on the Nyquist curve — point-wise for all frequencies

$$|\delta(i\omega)| < \left| \frac{1 + L(i\omega)}{L(i\omega)} \right| = \frac{1}{|T(i\omega)|}, \quad \forall \omega \geq 0. \tag{3}$$

- **Condition** (3) **is one of the reasons why feedback systems work so well in practice.**
  - The models used to design control systems are often simplified, and the properties of a process may change during operation.
  - Condition (3) implies that the closed loop system will at least be stable for substantial variations in the process dynamics

The peak value of the complimentary sensitivity:

$$M_t = \max_{\omega} |T(i\omega)| := \left\| \frac{PC}{1 + PC} \right\|_{\infty}$$

- Condition (3) becomes $|\delta(i\omega)| < 1/M_t, \forall \omega \geq 0.$
- Reasonable values of M_t are from 1.2 to 2.

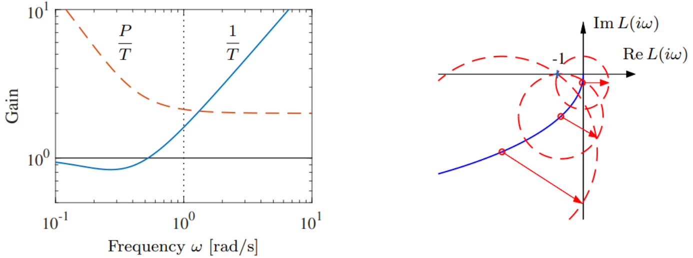
Example

Example (Example 13.7: Cruise Control)

The model of the car in the fourth gear at speed $20m/s$ is

$$P(s) = \frac{1.32}{s + 0.01}$$

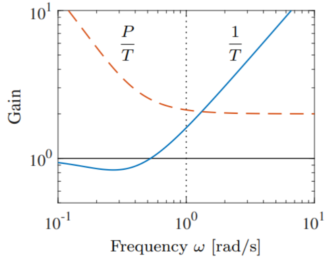
- Consider a PI controller with gains $k_p = 0.5$ and $k_i = 0.1$.



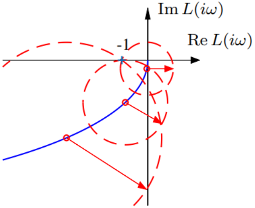
(a) Bounds on process uncertainty (b) Nyquist plot representation of bounds

Figure: Robustness of a cruise controller

Example



(a) Bounds on process uncertainty

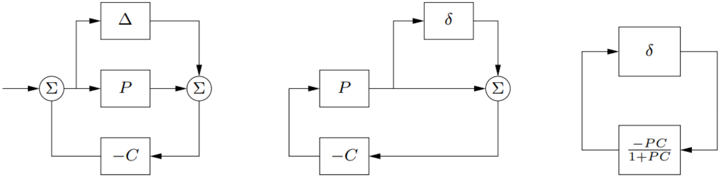


(b) Nyquist plot representation of bounds

Some observations:

- Moderately small uncertainties are required only around the **gain crossover frequencies**,
- but large uncertainties can be permitted at higher and lower frequencies.
- A simple model that describes the process dynamics well around the crossover frequency is often sufficient for design

Other robustness conditions



(a) Additive uncertainty (b) Multiplicative uncertainty (c) Small gain

Figure: Illustration of robustness to process perturbations[1]

Table 13.1: Conditions for robust stability for different types of uncertainty.

Process	Uncertainty Type	Robust Stability
$P + \Delta$	Additive	$\|CS\Delta\|_\infty < 1$
$P(1+\delta)$	Multiplicative	$\|T\delta\|_\infty < 1$
$P/(1+\Delta_{\text{fb}} \cdot P)$	Feedback	$\|PS\Delta_{\text{fb}}\|_\infty < 1$

[1] The details of these sufficient conditions are not required in this class.

Outline

Robust stability (continue from Lecture 24)

Fundamental limits - System design considerations

Summary

System design

The *initial design of a system* can have a significant impact on the ability to use feedback to provide **robustness** and **performance improvements**.

- It is particularly important to recognize fundamental limits in the performance of feedback systems early in the design process.
- Awareness of the limits and **co-design** of the process and the controller are good to avoid potential difficulties both for system and control designers.

Examples:

- We may expect that a system with **time delays** cannot admit fast control because control actions are delayed.
- It seems reasonable that **unstable systems will require fast controllers**, which will depend on the **bandwidth** of sensors and actuators.
- These limits are caused by properties of the system dynamics and can often be captured by conditions on the **poles and zeros** of the process

System design

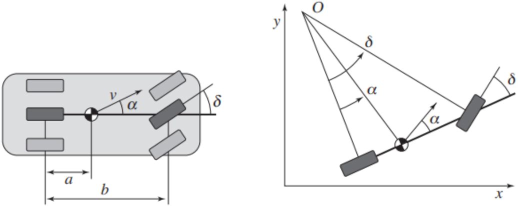
The freedom for the control designer depends very much on the situation

- **Extreme 1 (limited freedom)**: a process with given sensors and actuators and his or her task is to design a suitable controller
  - Even further, you may be given with an existing control loop, and your task is to retrofit the system.
- **Extreme 2 (significant freedom)**: You can choose sensors/actuators
  - **Co-design** the location and characteristics of sensors, actuators, and controller simultaneously.
  - However, you may have budget limits.

Performance limits due to dynamics and limits on actuation power/rate.

- **Dynamics limitations**: captured by time delays and poles and zeros in the right half-plane.
  - Time delays are easy to understand.
  - A less obvious case is that a process with a right half-plane pole/zero pair cannot be controlled robustly if the pole is close to the zero (details are not required in this class).
- **Restriction in actuation**: captured by actuation power and rates.

Example: Vehicle steering



(a) Overhead view (b) Bicycle model

Figure: Vehicle steering dynamics. The left figure shows an overhead view of a vehicle with four wheels. The wheelbase is b.

- The center of mass at a distance a forward of the rear wheels.
- Approximation with a single front wheel and a single rear wheel — an abstraction called the **bicycle model**.
- The steering angle is δ and the velocity at the center of mass has the angle α relative the length axis of the vehicle.
- The position is given by (x, y) and the orientation (heading) by θ; For ODE modeling, see Example 3.11

Example: Vehicle steering

Example (Frequency modeling for vehicle steering)

The transfer function from steering angle δ to lateral position y is

$$P(s) = \frac{av_0 s + v_0^2}{bs^2}$$

- v_0 is the velocity of the vehicle and $a, b > 0$
- The transfer function has a zero

$$s = -\frac{v_0}{a}.$$

  - In normal (forward) driving this zero is in the left half-plane,
  - but it is in the right half-plane when driving in reverse ($v_0 < 0$).
- The unit step response is $y(t) = \frac{av_0}{b} t + \frac{v_0^2}{2b} t^2$
  - The lateral position thus begins to respond immediately to a steering command as an integrator.
  - If $v_0 < 0$ (reverse steering), the initial $y(t)$ is in the wrong direction!!
  - This behavior is representative for **non-minimum phase systems** (called an **inverse response**).

Example: Vehicle steering

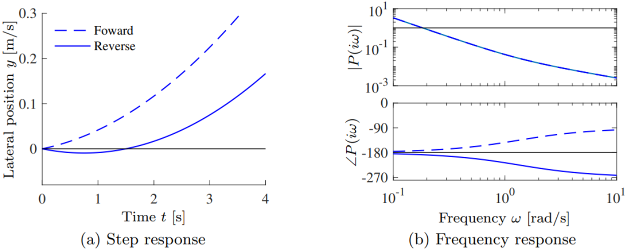
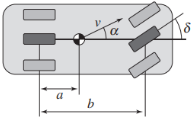


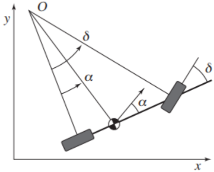
Figure: Vehicle steering for driving in reverse. (a) Step responses; (b) frequency responses

- The step response for forward and reverse driving is shown above.
- The parameters are a $a = 1.5m$, $b = 3m$, $v_0 = 2m/s$ for forward driving, and $v_0 = -2m/s$ for reverse driving.
- When driving in reverse, there is an initial motion of the center of mass in the opposite direction
- there is **A DELAY** before the car begins to move in the desired manner.

Example: Vehicle steering



(a) Overhead view



(b) Bicycle model

The existence of the right half-plane zero can be removed

- if we choose to measure the location of the vehicle by the position of the center of the rear wheels instead of the center of mass

$$a = 0, \qquad P(s) = \frac{v_0^2}{bs}$$

- This is easily implemented by calibrating the position sensor for the vehicle
- This choice of "sensor" is subject to calibration errors ϵ and this can lead to a zero of the process transfer function at v_0/ϵ
- This is called a **"fast" zero** and its impact is relatively minor.

Poles and Zeros

- The poles of a system depend on the intrinsic dynamics of the system.
- They represent the modes of the system and they are given by the eigenvalues of the dynamics matrix A of the linearized model.
  - For example, we have the initial response to $\dot{x} = Ax$

$$x_i(t) = e^{\lambda_i t} x_i(0).$$

Sensors and actuators have no effect on the poles: **the only way to change poles is by feedback or by redesign of the process.**

- However, the zeros of a system depend on how the sensors and actuators are connected to the process.
- Zeros can thus be changed by **moving or adding sensors and actuators**, which is often simpler than redesigning the process dynamics

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Robust stability (continue from Lecture 24)

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Summary

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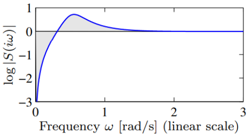
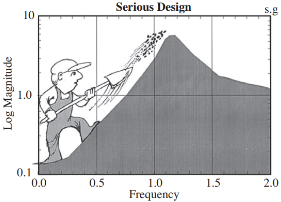
- **Performance limits** due to process dynamics and limits on actuation power/rate.
  - **Dynamics limitations**: captured by time delays and poles and zeros in the right half-plane.
    - Sensors and actuators have no effect on the poles: **the only way to change poles is by feedback or by redesign of the process.**
    - However, the zeros of a system depend on how the sensors and actuators are connected to the process.
  - **Restriction in actuation**: captured by actuation power and rates.
- **Waterbed Effect** — Bode's integral formula (will be covered in Lecture 26)

$$\int_0^\infty \log |S(i\omega)| d\omega = \int_0^\infty \log \frac{1}{|1 + L(i\omega)|} d\omega = \pi \sum p_k$$



(a) Bode integral formula



(b) Control design process