ECE 171A: Linear Control System Theory
Lecture 5: System Modeling (II)

Yang Zheng

Assistant Professor, ECE, UCSD

April 06, 2022

Reading materials: Ch 3.3, 3.4, Ch 4.7

System modeling

General nonlinear system

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

Linear time-invariant (LTI) system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$

$x \in \mathbb{R}^n$: state; $y \in \mathbb{R}^p$: output; $u \in \mathbb{R}^m$: input

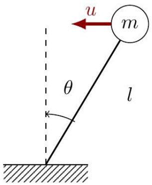
- **State** captures effects of the past
  - Physical quantities that determines future evolution;
- **Inputs** describe external excitation
  - Inputs are extrinsic to the system dynamics (externally specified);
- **Dynamics** describe state evolution
  - Update rule for system state; Function of current state and any external inputs;
- **Outputs** describe measured quantities
  - Outputs are function of state and inputs; not independent variables;

All models are wrong, but some are useful

Outline

Outline

Inverted pendulum



$m =$ mass

$l =$ length

$u =$ external force

$\theta =$ angle

- **Torque:** $T = mgl\sin\theta - ul\cos\theta$.
- **Moment of inertia:** $J = ml^2$.
- **Newton's law:**

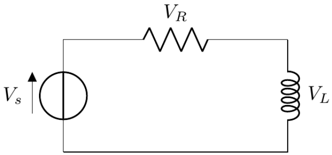
$$ml^2\ddot{\theta} = mgl\sin\theta - ul\cos\theta.$$

- **State-space model (nonlinear)**

$$\begin{aligned} x_1(t) &= \theta(t), \\ x_2(t) &= \dot{\theta}(t), \end{aligned} \qquad \Rightarrow \qquad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ \dfrac{mgl\sin\theta - ul\cos\theta}{ml^2} \end{bmatrix},$$

and $y = \theta(t)$.

RL Circuit



$$
\begin{aligned}
R :&\quad \text{Resistance}\\
L :&\quad \text{Inductance}\\
V_R = R \cdot I :&\quad \text{Resistor}\\
V_L = L \cdot \dot{I} :&\quad \text{Inductor}
\end{aligned}
$$

- **Kirchhoff's voltage law:**

$$V_S - V_R - V_L = 0.$$

- **Combining:**

$$L \cdot \dot{I} = V_S - V_R = V_S - RI$$

- **State-space model:** Let $x = I, u = V_S, y = V_R$, we have

$$
\begin{aligned}
\dot{x} &= -\frac{R}{L}x + \frac{1}{L}u \qquad \leftarrow \qquad \text{first-order ODE}\\
y &= Rx.
\end{aligned}
$$

Outline

Difference equations

In some situations, it is more natural to describe the evolution of a system at discrete instants of time rather than continuously in time

$\rightarrow$ **discrete-time systems**

- **General dynamics**

$$\begin{aligned} x[k+1] &= f(x[k], u[k]), \\ y[k] &= h(x[k], u[k]). \end{aligned}$$

  - $x \in \mathbb{R}^n$: state vector;
  - $u \in \mathbb{R}^n$: input vector;
  - $y \in \mathbb{R}^n$: output vector;

- **Linear difference equation**

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k], \\ y[k] &= Cx[k] + Du[k]. \end{aligned}$$

Note that the matrices A, B, C, D determine the response of this system:

Time evolution

Linear difference equation

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k], \\ y[k] &= Cx[k] + Du[k]. \end{aligned}$$

- At time $k = 1$

$$x[1] = Ax[0] + Bu[0], \qquad \begin{aligned} y[1] &= Cx[1] + Du[1] \\ &= CAx[0] + CBu[0] + Du[1]. \end{aligned}$$

- At time $k = 2$

$$\begin{aligned} x[2] &= Ax[1] + Bu[1] = A^2x[0] + ABu[0] + Bu[1], \\ y[2] &= Cx[2] + Du[2] \\ &= CA^2x[0] + CABu[0] + CBu[1] + Du[2]. \end{aligned}$$

- At time k (via **repeated substitution**)

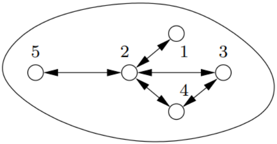
$$x[k] = A^k x[0] + \sum_{t=0}^{k-1} A^{k-t-1} Bu[t]$$

$$y[k] = CA^k x[0] + \sum_{t=0}^{k-1} CA^{k-t-1} Bu[t] + Du[k],$$

Consensus protocol

Goal: compute the average value of a set of numbers that are locally available to individual agents; *Applications*

- monitoring environment conditions in a region using multiple sensors
- monitoring movement of animals or vehicles
- monitoring the resource loading across a group of computers.



Adjacency matrix

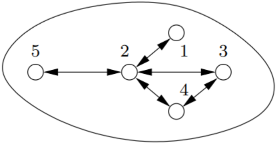
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

- $x_i \in \mathbb{R}$ denotes the state of the ith sensor
- update rule (dynamics)

$$x_i[k+1] = x_i[k] + \gamma \sum_{j \in \mathcal{N}_i} (x_j[k] - x_i[k]),$$

where $\mathcal{N}_i$ represents the set of neighbors of a node i.

Consensus protocol



Adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

- Collective dynamics

$$x[k+1] = x[k] - \gamma(D - A)x[k],$$

where D is a diagonal matrix with entries being the number of neighbors of each node.

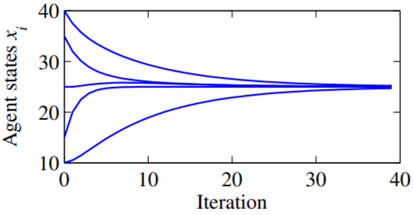


Figure: Consensus convergence

Predator-prey dynamics

Predator–prey problem: an ecological system in which we have two species, one of which feeds on the other.

- This type of system has been studied for decades and is known to exhibit interesting dynamics, e.g., *oscillation*.

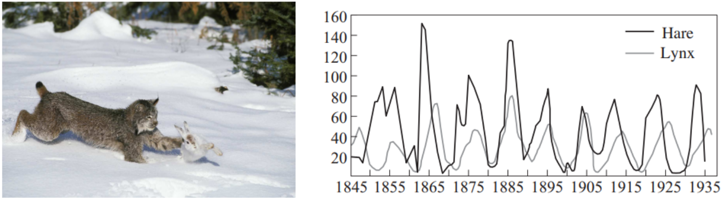


Figure 3.7: Predator versus prey. The photograph on the left shows a Canadian lynx and a snowshoe hare, the lynx's primary prey. The graph on the right shows the populations of hares and lynxes between 1845 and 1935 in a section of the Canadian Rockies [Mac37]. The data were collected on an annual basis over a period of 90 years. (Photograph copyright Tom and Pat Leeson.)

Predator-prey dynamics

A simple discrete-time model

- Predator - lynxes; Prey - hares

H : represent the population of hares;

L : represent the population of lynxes;

k : be the discrete-time index (*e.g.*, the month number).

- A simple model can be formulated as

$$
\begin{aligned}
H[k+1] &= H[k] + b_{\mathrm{h}}(u)H[k] - aL[k]H[k], \\
L[k+1] &= L[k] - d_{\mathrm{l}}L[k] \qquad + cL[k]H[k],
\end{aligned}
$$

  - b_{h} is the hare birth rate per unite period and is a function of the food supply u;
  - d_{l} is the lynx mortality rate;
  - a and c are the interaction coefficients;
  - $aL[k]H[k]$ is the rate of predation;
  - $cL[k]H[k]$ is the growth rate of the lynxes;

Predator-prey dynamics

Numerical simulation

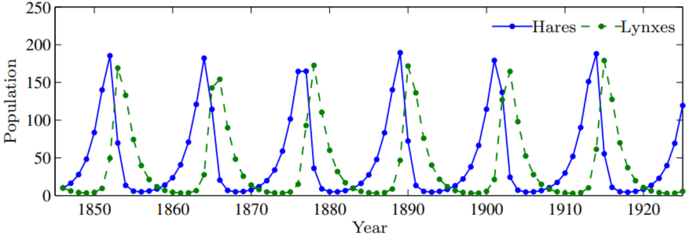


Figure 3.8: Discrete-time simulation of the predator–prey model (3.13). Using the parameters $a = c = 0.014$, $b_h(u) = 0.6$, and $d_l = 0.7$ in equation (3.13), the period and magnitude of the lynx and hare population cycles approximately match the data in Figure 3.7.

- The simulation details are different from the experimental data (expected)
- We see qualitatively similar trends
- Hence we can use the model to help explore the dynamics of the system

Outline

Inverted pendulum and RL-circuit

Difference equations

Population dynamics, Block diagrams

Summary

Population dynamics

Population growth is a complex dynamic process that involves the interaction of one or more species with their environment and the larger ecosystem.

- Predator-prey model
- **Logistic Growth model**

- Let x be the population of a species at time t

$$\frac{dx}{dt} = bx - dx = (b-d)x = rx, \qquad x \geq 0$$

  where birth rate b and mortality rate d are parameters.

- Exponential increase if $b > d$; or exponential decrease if $b < d$
- *A more realistic model*: the birth rate decreases when x is large

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right), \quad x \geq 0$$

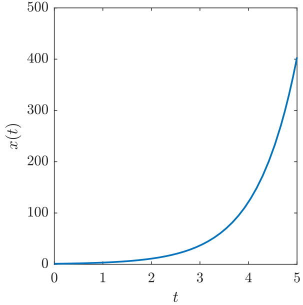
  where k is the *carrying capacity* of the environment — **Logistic Growth model**

Population dynamics

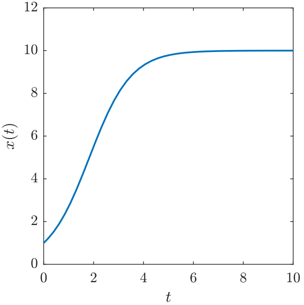
A more realistic model: the birth rate decreases when x is large

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right), \quad x \geq 0$$

where k is the *carrying capacity* of the environment — **Logistic Growth model**



(a) Exponential growth - $r = 1.2$



(b) Logistic Growth model with parameters $r = 1.2, k = 10$

Block diagrams

A special graphical representation called a **block diagram** has been developed in control engineering.

- Emphasize the information flow and to hide details of the system.

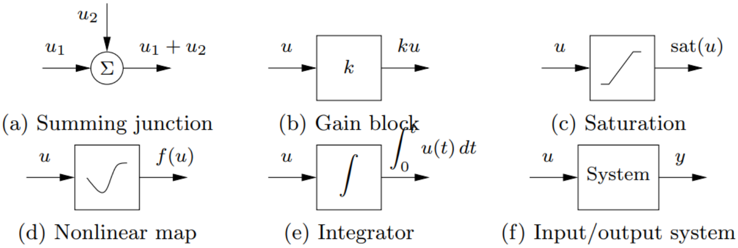


Figure: Standard block diagram elements. The arrows indicate the the inputs and outputs of each element, with the mathematical operation corresponding to the block labeled at the output.

Outline

Summary

- $x \in \mathbb{R}^n$: state; $y \in \mathbb{R}^p$: output; $u \in \mathbb{R}^m$: input
- Continuous-time

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du. \end{aligned}$$

- Discrete-time (e.g., consensus protocol, predator-prey dynamics)

$$\begin{aligned} x[k+1] &= f(x[k], u[k]), \\ y[k] &= h(x[k], u[k]). \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} x[k+1] &= Ax[k] + Bu[k], \\ y[k] &= Cx[k] + Du[k]. \end{aligned}$$

- **Block diagrams**: Emphasize the information flow and to hide details of the system.