ECE 171A: Linear Control System Theory
Lecture 7: Equilibrium and Stability

Yang Zheng

Assistant Professor, ECE, UCSD

April 11, 2022

Reading materials: Ch 5.3

Outline

Outline

Equilibrium points

An **equilibrium** point of a dynamical system represents a *stationary* condition for the dynamics.

- An **equilibrium point** for a dynamical system

$$\dot{x} = F(x),$$

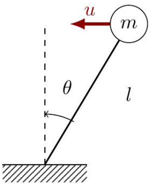
  is a state x_{e} such that $F(x_{\mathrm{e}}) = 0$.

- If a dynamical system has an initial condition $x(0) = x_{\mathrm{e}}$, then it will stay at the equilibrium point: $x(0) = x_{\mathrm{e}}$ for all $t \geq 0$ $(t_0 = 0)$.

- Equilibrium points are important since they correspond to **constant operating conditions**.

- A dynamical system can have zero, one, or more equilibrium points.

Example: Inverted pendulum



$m =$ mass

$l =$ length

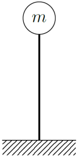
$u =$ external force

$\theta =$ angle

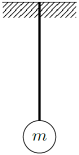
- Assume no external force — open-loop dynamics, $u = 0$

$$x_1(t) = \theta(t), \quad x_2(t) = \dot{\theta}(t), \quad \Rightarrow \quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ \dfrac{g \sin \theta}{l} \end{bmatrix} \quad \Rightarrow \quad x_{\mathrm{e}} = \begin{bmatrix} \pm n\pi \\ 0 \end{bmatrix}, n = 0, 1, 2 \ldots$$

Equilibrium 1 (unstable)



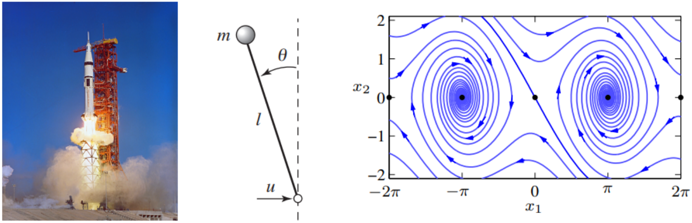
Equilibrium 2 (stable)



Example

- The equilibrium points are

$$x_e = \begin{bmatrix} \pm n\pi \\ 0 \end{bmatrix}, n = 0, 1, 2 \ldots$$



(a) (b) (c)

Figure 5.4: Equilibrium points for an inverted pendulum. An inverted pendulum is a model for a class of balance systems in which we wish to keep a system upright, such as a rocket (a). Using a simplified model of an inverted pendulum (b), we can develop a phase portrait that shows the dynamics of the system (c). The system has multiple equilibrium points, marked by the solid dots along the $x_2 = 0$ line.

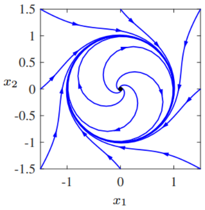
Limit cycles

Apart from equilibrium points, nonlinear systems can also exhibit *stationary periodic solutions* — **Limit cycles**.

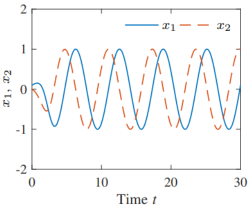
- This is of great practical value in generating sinusoidally varying voltages in power systems or in generating periodic signals for animal locomotion.

- Consider an electronic oscillator with dynamics

$$\dot{x}_1 = x_2 + x_1(1 - x_1^2 - x_2^2), \qquad \dot{x}_2 = -x_1 + x_2(1 - x_1^2 - x_2^2)$$



(a)



(b)

- The solutions in the phase plane converge to a circular trajectory.
- In the time domain, this corresponds to an oscillatory solution.

Outline

Stability of a solution

Stability of a solution of $\dot{x} = F(x)$: whether or not solutions nearby the solution remain close, get closer, or move further away.

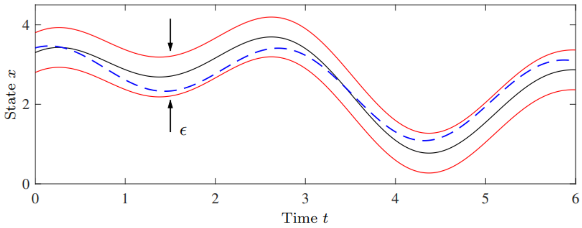


Figure 5.6: Illustration of Lyapunov's concept of a stable solution. The solution represented by the solid line is stable if we can guarantee that all solutions remain within a tube of diameter ϵ by choosing initial conditions sufficiently close the solution.

- Let $x(t; a)$ be a solution with initial condition a
- $x(t; a)$ is stable if for all $\epsilon > 0$, there exists a $\delta > 0$, such that

$$\|b - a\| < \delta \qquad \Rightarrow \qquad \|x(t; b) - x(t; a)\| < \epsilon, \text{ for all } t > 0.$$

Stability of equilibrium points

An important special case is when the solution $x(t; a) = x_e$ is an equilibrium solution. In this case the condition for stability becomes

$$\|x(0) - x_e\| < \delta \qquad \Rightarrow \qquad \|x(t) - x_e\| < \epsilon, \text{ for all } t > 0.$$

- **Stable**: we start near the equilibrium point, we stay near the equilibrium point — *stability in the sense of Lyapunov*

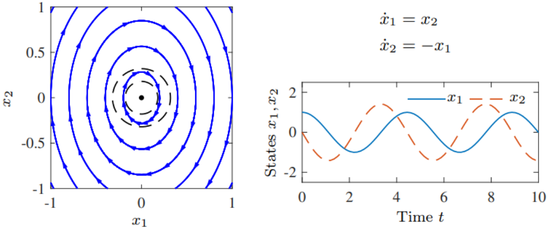


Figure: Phase portrait and time domain simulation: The equilibrium point x_e at the origin is stable since all trajectories that start near x_e stay near x_e

Asymptotically stable equilibrium

- **Asymptotically stable**: the equilibrium point is stable + all nearby trajectories converge to it

$$\|x(0) - x_{\mathrm{e}}\| < \delta \qquad \Rightarrow \qquad \|x(t) - x_{\mathrm{e}}\| < \epsilon \quad \text{and} \quad \lim_{t\to\infty} x(t) = x_{\mathrm{e}}.$$

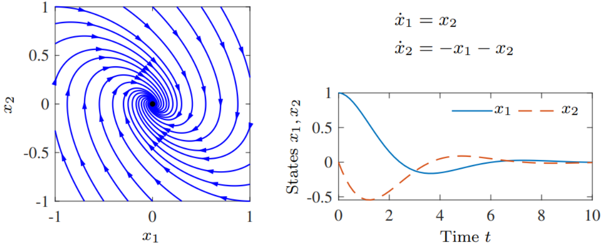


Figure: Phase portrait and time domain simulation: The equilibrium point x_{e} at the origin is asymptotically stable since the trajectories converge to this point as $t \to \infty$

Unstable equilibrium

- **Unstable**: the equilibrium point is unstable if it is not stable

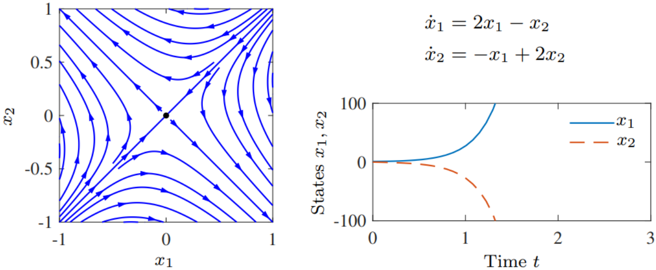
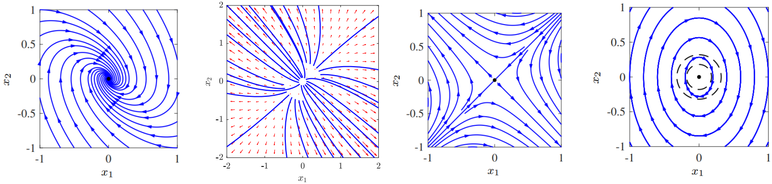


Figure: Phase portrait and time domain simulation: The equilibrium point x_e at the origin is unstable since not all trajectories that start near x_e stay near x_e . The sample trajectory on the right shows that the trajectories very quickly depart from zero.

Sink, Source, Saddle

For *planar dynamical systems*, equilibrium points have been assigned names based on their stability type.

- An *asymptotically stable equilibrium point* is called a **sink** or sometimes an **attractor**.
- An *unstable equilibrium point* can be either a **source**, if all trajectories lead away from the equilibrium point, or a **saddle**, if some trajectories lead to the equilibrium point and others move away
- An equilibrium point that is *stable but not asymptotically stable* (i.e., neutrally stable) is called a **center**



(a) Sink (b) Source (c) Saddle (d) Center

Outline

Stability

A linear dynamical system has the form

$$\dot{x} = Ax, \qquad x(0) = x_0.$$

- For a linear system, the stability of the equilibrium point at the origin can be determined from the eigenvalues of A

$$\lambda(A) = \{s \in \mathbb{C} \mid \det(sI - A) = 0\}.$$

Example

Consider a simple 2nd-order system with fully decoupled dynamics

$$\frac{dx}{dt} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- It can be written as $\qquad \dot{x}_1 = \lambda_1 x_1, \qquad \dot{x}_2 = \lambda_2 x_2$
- Its solution is

$$x_i = e^{\lambda_i t} x_i(0), i = 1, 2.$$

- $x_e = 0$ is stable if $\lambda_i \leq 0, i = 1, 2$, and asymptotically stable if $\lambda_i < 0, i = 1, 2$.

Stability

Theorem (Stability of a linear system)

The system $\dot{x} = Ax$ is

- **asymptotically stable** *if and only if all eigenvalues of A have a strictly negative real part, i.e.,* $\mathrm{Re}(\lambda_i) < 0$
- **unstable** *if any eigenvalues A has a strictly positive real part.*

Remark: If $\mathrm{Re}(\lambda_i) \leq 0, i = 1, \dots, n$ and some $\mathrm{Re}(\lambda_i) = 0$, the stability conditions are more complicated, which is beyond the scope of this class.

Example (Unstable systems)

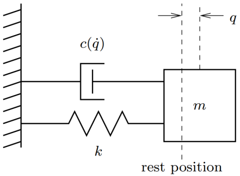
Consider the system $\ddot{q} = 0$. It can be written in state-space form as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

- The system has eigenvalues $\lambda = 0$, but the solutions are not bounded

$$x_1(t) = x_1(0) + x_2(0)t, \qquad x_2(t) = x_2(0).$$

Example: spring-mass system



System model: find the relation between the force F and the position q

$$m\ddot{q} + c\dot{q} + kq = F.$$

Suppose $F = 0$ and analyze the stability of this system.

- State-space model is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Compute its eigenvalues

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & -1 \\ \frac{k}{m} & \lambda + \frac{c}{m} \end{bmatrix}\right) = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

- The eigenvalues have negative real parts

$$\lambda_1 = \frac{-\frac{c}{m} + \sqrt{\left(\frac{c}{m}\right)^2 - \frac{4kc}{m}}}{2}, \qquad \lambda_2 = \frac{-\frac{c}{m} - \sqrt{\left(\frac{c}{m}\right)^2 - \frac{4kc}{m}}}{2}$$

as long as $c > 0$ (damper). The system is *asymptotically stable*.

Routh–Hurwitz Criterion

- It can often be difficult to analytically compute the roots of a high-order polynomial.
- **The Routh–Hurwitz criterion** is a stability criterion that does not require explicit calculation of the roots, because it gives conditions in terms of the coefficients of the **characteristic polynomial** – further on this later.

Example (Second-order systems)

Consider a second-order polynomial

$$a\lambda^2 + b\lambda + c = 0$$

- The Routh table is

$$\begin{array}{lrr} \lambda^2 & a & c \\ \lambda^1 & b & 0 \\ \lambda^0 & -\dfrac{1}{b}(a \times 0 - bc) = c & 0 \end{array}$$

- The eigenvalues has strictly negative real parts if and only if the first column of the Routh table is non-zero and has no sign changes.

Routh–Hurwitz Criterion

Example (Third-order systems)

Consider a third-order polynomial

$$a\lambda^3 + b\lambda^2 + c\lambda + d = 0$$

- The Routh table is

$$\begin{array}{lrr}
\lambda^3 & a & c \\
\lambda^2 & b & d \\
\lambda^1 & -\frac{1}{b}(ad - bc) & 0 \\
\lambda^0 & d & 0
\end{array}$$

- The eigenvalues has strictly negative real parts if and only if the first column of the Routh table is non-zero and has no sign changes.
- If $a > 0$, then a sufficient and necessary condition for stability (all eigenvalues have strictly negative real parts) is

$$a > 0, \qquad b > 0, \qquad bc > ad, \qquad d > 0$$

Routh table

- $p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0$

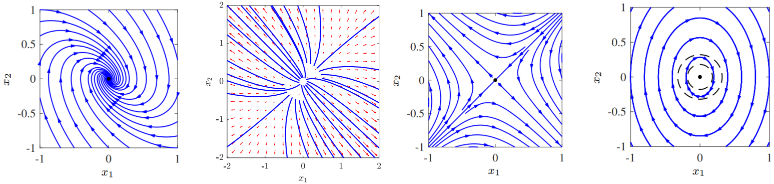
λ^n	a_n	a_{n-2}	a_{n-4}	$\cdots$	a_0
λ^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$	0
λ^{n-2}	$b_{n-1} = -\frac{\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}}$	$b_{n-3} = -\frac{\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{a_{n-1}}$	b_{n-5}	$\cdots$	0
λ^{n-3}	$c_{n-1} = -\frac{\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}}{b_{n-1}}$	$c_{n-3} = -\frac{\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_{n-1} & b_{n-5} \end{vmatrix}}{b_{n-1}}$	c_{n-5}	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$
λ^0	a_0	0	0	$\cdots$	0

- Any row can be multiplied by a positive constant without changing the result

Outline

Summary

- An **equilibrium** point of a dynamical system represents a *stationary* condition for the dynamics.
- Stable, asymptotically stable, unstable — sink, source, saddle, center



(a) Sink (b) Source (c) Saddle (d) Center

- Stability of linear systems
  - Eigenvalue test
  - **Routh–Hurwitz** Criterion