ECE 171A: Linear Control System Theory
Lecture 9: Review

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Announcements

- HW2 due by tonight
- Midterm exam (I) — in class, April 22 (Friday in Week 4)
  - **Scope**: Lectures 1 - 10, HW1 - HW3, DI 1-4; (Reading materials in the textbook)
  - Closed book, closed notes, closed external links.
  - **Come on time** (1 or 2 minutes early if you can; we will start at 9:00 am promptly)
  - No MATLAB is required. No graphing calculators are permitted. A basic arithmetic calculator is allowed.
  - The exams must be done in a blue book. Bring a blue book with you.
  - **No collaboration and discussions are allowed**. It is dishonest to cheat on exams. Instances of academic dishonesty will be referred to the Office of Student Conduct for adjudication. *You don't want to take a risk for such a small thing*.

Outline

Review: Lectures 1 - 8

Two Exercises

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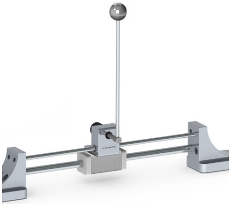
Review: Lectures 1 - 8

Two Exercises

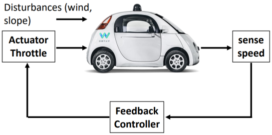
Lecture 1 - Overview of control systems

A **control system** is an interconnection of two or more dynamical systems that provides a **desired response**.

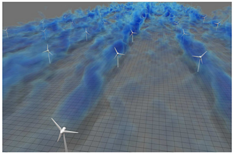
- Control is to modify the inputs to the plant to produce a **desired output**.



(a) Inverted Pendulum



(b) Cruise control



(c) Wind farm

- Feedforward control vs. feedback control
- Two live experiments
- **Feedback control = Sensing + Computation + Actuation**

Lecture 2 - ODEs and the first control example

Review on ODEs

- An n**th-order linear ordinary differential equation (ODE)** is:

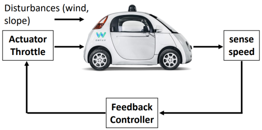
$$\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_1\frac{d}{dt}y(t) + a_0 y(t) = u(t)$$

- First-order matrix ODE

$$\dot{x} = Ax(t) + Bu(t)$$

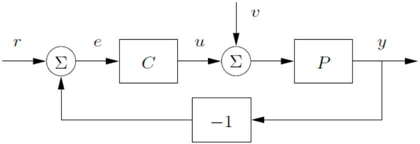
Cruise control

P control $F_{\text{engine}} = K_{\text{p}} e(t)$
I control $F_{\text{engine}} = K_{\text{i}} \int_0^t e(t)dt$
D control $F_{\text{engine}} = K_{\text{d}} \frac{d}{dt} e(t)$



Feedback control $=$ **Sensing** $+$ **Computation** $+$ **Actuation**

Lecture 3 - Feedback principles



- We have considered static plant dynamics with analytical solutions

$$y = \mathrm{sat}(x) = \begin{cases} -1 & \text{if } x \leq -1 \\ x & \text{if } |x| < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

- and a simple dynamical model with numerical simulations
- to illustrate several fundamental properties of feedback

- Disturbance attenuation
- Reference signal tracking
- Robustness to uncertainty
- Shaping of dynamical behavior

Lecture 4/5 - System modeling

A model is a *mathematical representation* of a physical, biological, or information system.

- Models allow us to reason about a system and make predictions about how a system will behave.
- The model we choose depends on the questions we wish to answer, and so there may be multiple models for a single dynamical system.
- It is important to keep in mind that all models are an approximation of the underlying system.

The choice of state is not unique.

- There may be many choices of variables that can act as the state.
  - A trivial example: One can choose different units (scaling factors)
  - A less trivial example: One can take sums and differences of some variables.

Lecture 4/5 - System modeling

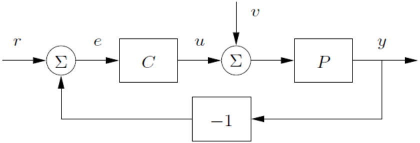
- $x \in \mathbb{R}^n$: state; $y \in \mathbb{R}^p$: output; $u \in \mathbb{R}^m$: input
- Continuous-time (e.g., speed control, inverted pendulum, spring-mass)

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du. \end{aligned}$$

- Discrete-time (e.g., consensus protocol, predator-prey dynamics)

$$\begin{aligned} x[k+1] &= f(x[k],u[k]), \\ y[k] &= h(x[k],u[k]). \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} x[k+1] &= Ax[k] + Bu[k], \\ y[k] &= Cx[k] + Du[k]. \end{aligned}$$

- **Block diagrams**: Emphasize the information flow and to hide details of the system.

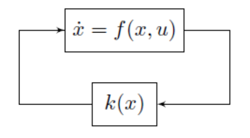


Lecture 6 - System solutions and Phase portraits

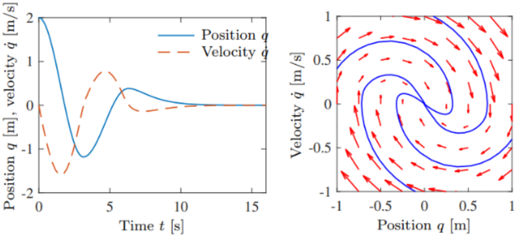
Closed-loop system: with $u = k(x)$

$$\dot{x}(t) = f(x, k(x)) := F(x).$$

Analytical or *Computational* solutions



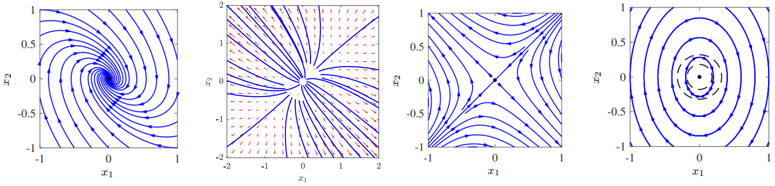
- **Solving differential equations**
- **Qualitative analysis**: phase portraits and time plot



(a) Time plot (b) Phase portrait

Lecture 7 - Equilibrium and stability

- An **equilibrium** point of a dynamical system represents a *stationary* condition for the dynamics.
- Stable, asymptotically stable, unstable; terminology like sink, source, saddle, center



(a) Sink (b) Source (c) Saddle (d) Center

- Stability of linear systems $\dot{x} = Ax$
  - Eigenvalue test
  - **Routh–Hurwitz** Criterion (will be reviewed again in week 5/6)

Lecture 8: Jacobian Linearization

Consider a nonlinear system $\dot{x} = F(x)$, with $x_e = 0$ as an equilibrium point. Let

$$A = \left.\frac{\partial F}{\partial x}\right|_{x_e=0}$$

- $x_e = 0$ is locally asymptotically stable if A is asymptotically stable or all eigenvalues of A have negative real parts.
- $x_e = 0$ is unstable if one or more of the eigenvalues of A has positive real part.

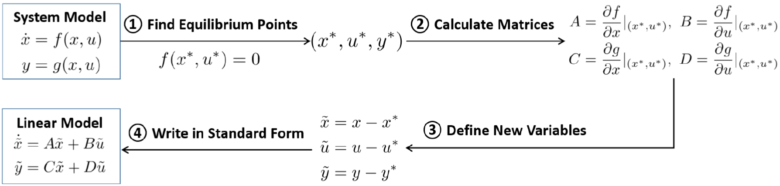


Figure: Model Linearization Procedure (Taken from Prof Na Li's ES 155)

Outline

Review: Lectures 1 - 8

Two Exercises

Exercise A: Linearization of nonlinear systems

Consider the following dynamical system

$$\dot{x}_1 = x_1 - x_2^2$$
$$\dot{x}_2 = 2x_1 - x_2^2 - x_1 x_2.$$

1. Determine the equilibrium point(s) of this system
2. Linearize the system around the equilibrium point(s)
3. Are the equilibrium points stable for the linearized system?

Solution:

- Let $f_1(x_1, x_2) = 0, f_2(x_1, x_2) = 0.$

$$\begin{aligned} x_1 - x_2^2 = 0 \\ 2x_1 - x_2^2 - x_1 x_2 = 0. \end{aligned} \quad \Rightarrow \begin{cases} x_1^* = 0 \\ x_2^* = 0 \end{cases} , \quad \begin{cases} x_1^* = 1 \\ x_2^* = 1 \end{cases}$$

- Define the dynamics

$$f_1(x_1, x_2) = x_1 - x_2^2$$
$$f_2(x_1, x_2) = 2x_1 - x_2^2 - x_1 x_2.$$

Exercise A: Linearization of nonlinear systems

- Compute their partial derivatives

$$\frac{\partial f_1}{\partial x_1} = 1, \quad \frac{\partial f_1}{\partial x_2} = -2x_2,$$
$$\frac{\partial f_2}{\partial x_1} = 2 - x_2, \quad \frac{\partial f_2}{\partial x_2} = -2x_2 - x_1$$

- For the first equilibrium $x_1^* = 0, x_2^* = 0$, there is no need to define new variables $\tilde{x}_1 = x_1 - x_1^* = x_1$. We have

$$\begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= 2x_1. \end{aligned} \qquad \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- For the second equilibrium point $x_1^* = 1, x_2^* = 1$, we define $\tilde{x}_1 = x_1 - 1$, $\tilde{x}_2 = x_2 - 1$, and we have

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_1 - 2\tilde{x}_2 \\ \dot{\tilde{x}}_2 &= \tilde{x}_1 - 3\tilde{x}_2. \end{aligned} \qquad \Rightarrow \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

Exercise A: Linearization of nonlinear systems

- For the first linearized system

$$\det(\lambda I - A) = \det\left(\begin{bmatrix}\lambda - 1 & 0\\ -2 & \lambda\end{bmatrix}\right) = (\lambda - 1)\lambda = 0$$

$$\Rightarrow \qquad \lambda_1 = 1, \lambda_2 = 0$$

This is unstable since $\lambda_1 = 1$ has positive real part.

- For the second linearized system

$$\det(\lambda I - A) = \det\left(\begin{bmatrix}\lambda - 1 & 2\\ -1 & \lambda + 3\end{bmatrix}\right) = \lambda^2 + 2\lambda - 1 = 0$$

$$\Rightarrow \qquad \lambda = -1 \pm \sqrt{2}$$

This is unstable since $\lambda_2 = -1 + \sqrt{2}$ has positive real part.

Exercise A: Linearization of nonlinear systems

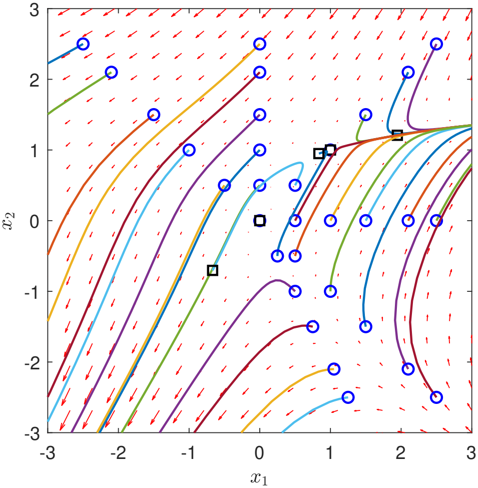


Figure: Phase portrait of the nonlinear system:
$\dot{x}_1 = x_1 - x_2^2, \dot{x}_2 = 2x_1 - x_2^2 - x_1 x_2.$

Example B: SpaceX rocket controller design

A rocket of mass m in vertical flight can be modeled by

$$\dot{h} = v$$
$$M\dot{v} = F - \frac{km}{h^2} - cv,$$

- $h > 0$ is the vertical distance away from the earth,
- v is the vertical velocity,
- F is the rocket engine thrust force (control input),
- $\frac{km}{h^2}$ represents the universal gravitation, and cv captures the friction.

Suppose $m = 1, k = 1, c = 1$; we let $x_1 = h$ and $x_2 = v$, and the output $y = h$, input $u = F$.

Question 1 - equilibrium point: Let $F^* = 1$. What is the equilibrium point of this system?

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{1}{x_1^2} - x_2 + u \end{aligned} \quad \Longrightarrow \quad \begin{cases} u^* = 1, x_1^* = 1, x_2^* = 0, \\ y^* = x_1^* = 1 \end{cases}$$

Example: SpaceX rocket controller design

Question 2 - Linearization: Linearize the system around the equilibrium point.

- **Step 1**: Write down the (possibly nonlinear) dynamics (step 0: obtain the equilibrium)

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, u) = x_2 \\ \dot{x}_2 = f_2(x_1, x_2, u) = -\dfrac{1}{x_1^2} - x_2 + u \end{cases}$$

- **Step 2**: compute their partial derivatives

$$\frac{\partial f_1}{\partial x_1} = 0, \quad \frac{\partial f_1}{\partial x_2} = 1, \quad \frac{\partial f_1}{\partial u} = 0,$$
$$\frac{\partial f_2}{\partial x_1} = \frac{2}{x_1^3}, \quad \frac{\partial f_2}{\partial x_2} = -1, \quad \frac{\partial f_2}{\partial u} = 1,$$

- **Step 3**: define new variables $\tilde{x} = x - x^*$, $\tilde{u} = u - u^*$, and $\tilde{y} = y - y^*$.
- **Step 4**: Finalize the linearized model

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{u}, \qquad \tilde{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}.$$

Example: SpaceX rocket controller design

Question 3 - controller design: SpaceX hover (and landing) test[1] - design a controller to stabilize the rocket at the equilibrium point.

$$\tilde{u} = K_1 \tilde{x}_1 + K_2 \tilde{x}_2.$$

If $K_1 = -2.25, K_2 = 0.5$, is the linearized system stable?

- **Step 1**: write down the closed-loop system

$$\tilde{u} = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \tilde{x}$$

$$\Rightarrow \quad \begin{aligned} \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 2 + K_1 & -1 + K_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \end{aligned}$$

[1] Videos: https://youtu.be/07Pm8ZY0XJI; https://youtu.be/1sJlFzUQVmY

Example: SpaceX rocket controller design

Question 3 - controller design: SpaceX hover (and landing) test[2] - design a controller to stabilize the rocket at the equilibrium point.

$$\tilde{u} = K_1 \tilde{x}_1 + K_2 \tilde{x}_2.$$

If $K_1 = -2.3125, K_2 = 0.5$, is the linearized system stable?

- **Step 1**: write down the closed-loop system

$$\dot{\tilde{x}} = \begin{bmatrix} 0 & 1 \\ 2 + K_1 & -1 + K_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.3125 & -0.5 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

- **Step 2**: analyze the eigenvalues of the closed-loop system

$$\begin{aligned} \det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & -1 \\ 0.3125 & \lambda + 0.5 \end{bmatrix} \right) &= \lambda^2 + 0.5\lambda + 0.3125 \\ &= (\lambda + 0.25)^2 + 0.25 \end{aligned}$$

its eigenvalues are $\lambda = -0.25 \pm 0.5i$, which is stable.

[2]Videos: `https://youtu.be/07Pm8ZY0XJI`; `https://youtu.be/1sJlFzUQVmY`