ECE 171A: Linear Control System Theory
Discussion 1: Review on ODEs (I)

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Outline

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First Order Linear Homogeneous ODEs

First Order Linear Nonhomogeneous ODEs

ode45 in Matlab

First-order linear homogeneous ODEs

In Lecture 2, we have discussed the first-order and second-order linear ODEs. Here, we review the methods for solving first-order linear ODEs.

A first-order **homogeneous** linear ODE is of the form

$$\dot{x}(t) = ax(t)$$

where $a \in \mathbb{R}$ is a constant, $\dot{x}(t)$ denotes the derivative of $x(t)$.

- The solution is
$$x(t) = e^{at}x(0).$$
(Recall the function with the first-order derivative being itself is $x(t) = e^t$.)
- We can easily verify it by observing
$$\dot{x}(t) = ae^{at}x(0) = ax(t),$$
$$x(0) = e^{a\times 0}x(0) = x(0).$$

Example 1: Stable system

Consider a first-order ODE

$$\dot{x}(t) + 2x(t) = 0, \quad x(0) = 1.$$

- We have $\dot{x}(t) = -2x(t)$ implying that

$$x(t) = x(0)e^{-2t} = e^{-2t},$$

- You can verify the answer by

$$\dot{x}(t) + 2x(t) = -2e^{-2t} + 2e^{-2t} = 0$$

  and $x(0) = 1$.

Example 2: stable system

Consider a first-order ODE

$$\dot{x}(t) + 2x(t) = 0, x(0) = 2$$

- It is similar to the previous problem except the different initial values are different.
- So the solution is

$$x(t) = x(0)e^{-2t} = 2e^{-2t}.$$

- You can verify the answer by

$$\dot{x}(t) + 2x(t) = -4e^{-2t} + 4e^{-2t} = 0$$

  and $x(0) = 2$.

Example 3: Unstable system

Consider a first-order ODE

$$\dot{x}(t) - 2x(t) = 0, \quad x(0) = 1$$

The solution is

$$x(t) = x(0)e^{2t} = e^{2t}.$$

You can verify the answer by

$$\dot{x}(t) - 2x(t) = 2e^{2t} - 2e^{2t} = 0$$

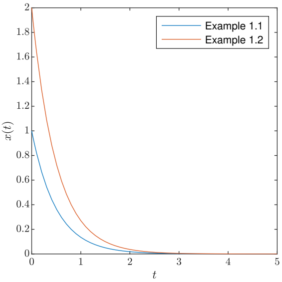
and $x(0) = 1$.

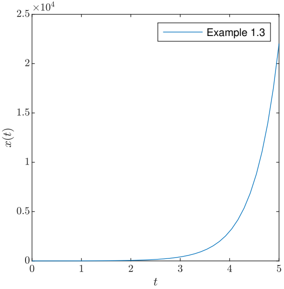
Stability

Stability can be judged by either their solutions or figures.

- The solutions of these stable systems are $c_1 e^{-2t}$ which all converge to 0;
- The solution of the unstable system is $c_2 e^{2t}$ which is unbounded.

Their solutions are shown in the following figures





Outline

Nonhomogeneous ODEs

We consider the general form of a first-order linear ODE:

$$\dot{x}(t) + cx(t) = u(t), \tag{1}$$

where $c \in \mathbb{R}$ is a given constant, and $u(t)$ is a given function.

- We multiply (1) by the integrating factor $\mu(t) = e^{ct}$.
- Since $\dot{\mu}(t) = c\mu(t)$, from (1), we have

$$\begin{aligned} \frac{d}{dt}(\mu(t)x(t)) &= \dot{\mu}(t)x(t) + \mu(t)\dot{x}(t) \\ &= \mu(t)(\dot{x}(t) + cx(t)) \\ &= \mu(t)u(t). \end{aligned}$$

- Now we denote $\mu(t)x(t) = g(t)$ and $\mu(t)u(t) = h(t)$. Then, we have

$$\dot{g}(t) = h(t).$$

Nonhomogeneous ODEs

Thus, we have

$$g(t) = \int h(t)dt + c_1$$

Let us go back to $x(t)$. This is the same as

$$\mu(t)x(t) = \int e^{ct}u(t)dt + c_1$$

Thus, the general solution is:

$$x(t) = e^{-ct}\left(\int e^{ct}u(t)dt + c_1\right),$$

where c is the same constant in (1), and c_1 is another constant to be determined from initial conditions.

Example 4: Stable system with positive input

Consider a first-order ODE

$$\dot{x}(t) + 2x(t) = 5, x(0) = 1$$

- The general solution is

$$\begin{aligned} x(t) &= e^{-2t} \left(\int 5e^{2t} dt + c_1 \right) \\ &= e^{-2t} \left(\frac{5}{2} e^{2t} + c_1 \right) = \frac{5}{2} + c_1 e^{-2t}, \end{aligned}$$

- Let us determine the constant c_1 from $x(0) = 1$.
- It is easy to see that $c_1 = -\frac{3}{2}$, leading to $x(0) = 1$.
- We can also verify the solution via

$$\dot{x}(t) + 2x(t) = -\frac{3}{2}(-2)e^{-2t} + 5 - 3e^{-2t} = 5.$$

Example 5: Stable system with negative input

Consider a first-order ODE

$$\dot{x}(t) + 2x(t) = -5, x(0) = 1$$

- The general solution is

$$\begin{aligned} x(t) &= e^{-2t}\left(\int -5e^{2t}dt + c_1\right) \\ &= e^{-2t}\left(-\frac{5}{2}e^{2t} + c_1\right) = -\frac{5}{2} + c_1 e^{-2t}, \end{aligned}$$

- Let us determine the constant c_1 from $x(0) = 1$.
- It is easy to see that $c_1 = \frac{7}{2}$, leading to $x(0) = 1$.
- We can also verify the solution via

$$\dot{x}(t) + 2x(t) = -\frac{7}{2}(-2)e^{-2t} - 5 - 7e^{-2t} = -5.$$

Example 6: System with polynomial input

Consider a first-order ODE

$$\dot{x}(t) + 2x(t) = t, x(0) = 1$$

- Using integration by parts, we have

$$\int te^{2t}dt = \frac{1}{2}te^{2t} - \frac{1}{2}\int e^{2t}dt = \frac{te^{2t}}{2} - \frac{e^{2t}}{4}$$

- The solution is

$$x(t) = e^{-2t}\left(\frac{te^{2t}}{2} - \frac{e^{2t}}{4} + c_1\right) = \frac{1}{2}t - \frac{1}{4} + c_1e^{-2t},$$

  where $c_1 = \frac{5}{4}$ satisfies $x(0) = 1$.
- We can check the answer by

$$\dot{x}(t) + 2x(t) = \frac{1}{2} + \frac{5}{4}(-2)e^{-2t} + t - \frac{1}{2} + \frac{5}{2}e^{-2t} = t.$$

Example 7: System with trigonometric input

Consider a first-order ODE

$$\dot{x}(t) + 2x(t) = \sin t, x(0) = 1$$

- Let us verify

$$x(t) = \frac{1}{5}(2\sin t - \cos t) + \frac{6}{5}e^{-2t},$$

  is a valid solution.
- First, we have

$$x(0) = -\frac{1}{5} + \frac{6}{5} = 1.$$

- Second, we have

$$\begin{aligned}\dot{x}(t) + 2x(t) &= \frac{1}{5}(2\cos t + \sin t) + \frac{6}{5}(-2)e^{-2t} \\ &\qquad + \frac{2}{5}(2\sin t - \cos t) + \frac{12}{5}e^{-2t} \\ &= \sin t.\end{aligned}$$

Review on Integration by parts

The integration by parts formula states:

- in the form of the indefinite integral

$$\int u(t)\dot{v}(t)dt = u(t)v(t) - \int \dot{u}(t)v(t)dt$$

- in the form of the definite integral

$$\begin{aligned}\int_a^b u(t)\dot{v}(t)dt &= [u(t)v(t)]_a^b - \int_a^b \dot{u}(t)v(t)dt \\ &= u(b)v(b) - u(a)v(a) - \int_a^b \dot{u}(t)v(t)dt\end{aligned}$$

In Examples 6 and 7, we have used the integration by parts to find antiderivative of te^{2t} and $e^{2t}\sin t$. In general, we let $\dot{v}(t) = e^{ct}$, i.e., $v(t) = \frac{1}{c}e^{ct}$.

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First Order Linear Nonhomogeneous ODEs

ode45 in Matlab

ode45 Matlab

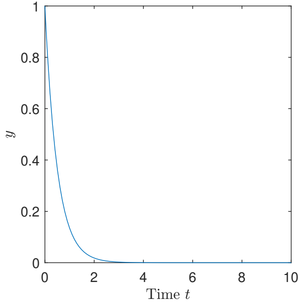
- Matlab ODE45 function:

```
[t,y] = ode45(odefun,tspan,y0)
```

- Many useful information can be found here
  https://www.mathworks.com/help/matlab/ref/ode45.html

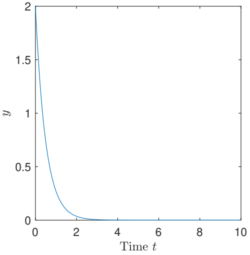
```
%----- Example 1 -------
%  \dot x = -2*x,
%   with x(0) = 1
%-----------------------

f1 = @(t,x)(-2*x); % vector field
[ts,ys] = ode45(f1,[0,10],1);
```

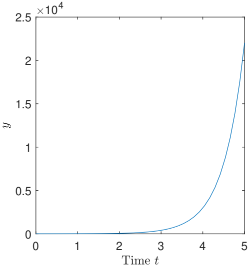


ode45 Matlab - Example 2 & 3

```
%----- Example 2 -------
%  \dot x = -2*x,
%   with x(0) = 2
%------------------------
f2 = @(t,x)(-2*x); % vector field
[ts,ys] = ode45(f2,[0,10],2);
```

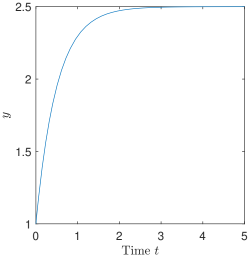



```
%------ Example 3 ------------
%  \dot x = 2*x,
%   with x(0) = 1
%-----------------------------
f3       = @(t,x)(2*x); % vector field
[ts,ys] = ode45(f3,[0,5],1);
```

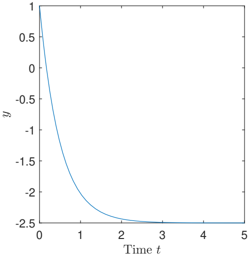


ode45 Matlab - Example 4 & 5

```
%------ Example 4 ------------
%  \dot x = -2*x + 5,
%   with x(0) = 1
%----------------------------
f4      = @(t,x)(-2*x+5); % vector field
[ts,ys] = ode45(f4,[0,5],1);
```

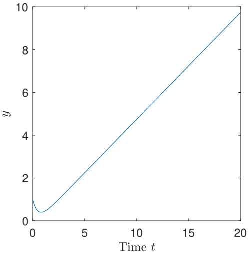



```
%------ Example 5 ------------
%  \dot x = -2*x - 5,
%   with x(0) = 1
%----------------------------
f5      = @(t,x)(-2*x -5); % vector field
[ts,ys] = ode45(f5,[0,5],1);
```



ode45 Matlab - Example 6 & 7

```
%------ Example 6 ------------
%  \dot x = -2*x + t,
%   with x(0) = 1
%----------------------------
f6      = @(t,x)(-2*x + t); % vector field
[ts,ys] = ode45(f6,[0,20],1);
```




```
%------ Example 7 ------------
%  \dot x = -2*x + sin t,
%   with x(0) = 1
%----------------------------
f7      = @(t,x)(-2*x + sin(t)); % vector fi
[ts,ys] = ode45(f7,[0,20],1);
```