ECE 171A: Linear Control System Theory

Discussion 2: Review on ODEs (II) - second-order ODEs and ode45

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Outline

Second-order linear ODEs

Second-order ODEs in matrix form

ode45 in Matlab

Outline

Second-order linear ODEs

A **homogeneous second-order linear ODEs** (with constant coefficients) is in the form

$$\ddot{z}(t) + a\dot{z}(t) + bz(t) = 0 \tag{1}$$

where $a \in \mathbb{R}$ and $b \in \mathbb{R}$ are constants.

- Candidate solutions: **exponential functions** e^{st}

Example

Consider $z(t) = e^{2t}$, we have

$$\begin{cases} \dot{z}(t) = 2e^{2t}, \\ \ddot{z}(t) = 4e^{2t} \end{cases}$$

$$\Rightarrow \qquad \ddot{z}(t) - \dot{z}(t) - 2z(t) = 4e^{2t} - 2e^{2t} - 2e^{2t} = 0$$

- $z(t) = e^{2t}$ is a particular solution to (1) with $a = -1, b = -2$.

Ansatz

- What we just did - guess a simple form of a solution and plug it in and see where that leads us - is a fairly common technique in the study of differential equations.
- Such a guess-solution is called an **ansatz**, a word of German origin (from Google, it means "approach" or "attempt") [1].

Definition (Ansatz)

An **educated** guess or an additional assumption made to help solve a problem, and which may later be verified to be part of the solution by its results[2].

> We will use them (ansatzes) in this discussion note.

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[1] `https://www.maths.usyd.edu.au/u/UG/IM/MATH2921/r/PDF/MatrixODEs.pdf`

[2] Taken from wikipedia `https://en.wikipedia.org/wiki/Ansatz`

Characteristic polynomial

We guess the solution to (1) is in the form of $z(t) = e^{st}$.

- Substituting the (1) by $z(t) = e^{st}$, we obtain that
$$\ddot{z}(t) + a\dot{z}(t) + bz(t) = s^2 e^{st} + ase^{st} + be^{st} = e^{st}(s^2 + as + b) = 0$$
- The value of s must satisfy
$$F(s) := s^2 + as + b = 0. \tag{2}$$
- $F(s)$ is called the **characteristic polynomial** associated with a homogeneous second-order ODE.

Solving the original ODE is reduced to solving an algebraic equation.
Three cases:

1. $F(s)$ has two distinct roots;
2. $F(s)$ has a double root;
3. $F(s)$ has a pair of complex roots;

Case 1: two distinct roots

If $a^2 - 4b > 0$, we have two distinct real roots s_1 and s_2,

$$s_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right),$$
$$s_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right).$$

- In this case, the general solution to (1) is

$$z(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

  where $c_1 \in \mathbb{R}, c_2 \in \mathbb{R}$ are real constants determined by the initial conditions.

Example 1

Example

Consider a second-order ODE

$$\ddot{z}(t) - 3\dot{z}(t) - 18z(t) = 0, \quad z(0) = 3, \quad \dot{z}(0) = 9 \tag{3}$$

- The characteristic polynomial is $s^2 - 3s - 18 = (s-6)(s+3)$ which has roots 6 and -3.
- Thus, the general solution is $z(t) = c_1 e^{6t} + c_2 e^{-3t}$ where c_1 and c_2 satisfy

$$\begin{cases} c_1 + c_2 = 3, \\ 6c_1 - 3c_2 = 9, \end{cases} \qquad \Longrightarrow \qquad c_1 = 2, c_2 = 1.$$

- The solution to (3) is $z(t) = 2e^{6t} + e^{-3t}$. We can verify this solution by

$$\begin{aligned} & \dot{z}(t) = 12e^{6t} - 3e^{-3t}, \ddot{z}(t) = 72e^{6t} + 9e^{-3t} \\ & \quad \ddot{z}(t) - 3\dot{z}(t) - 18z(t) \\ \Longrightarrow \qquad & = 72e^{6t} + 9e^{-3t} - 3(12e^{6t} - 3e^{-3t}) - 18(2e^{6t} + 1e^{-3t}) \\ & = 0. \end{aligned}$$

Case 2: a double root

If $a^2 - 4b = 0$, we have a double root, i.e.,

$$s_1 = s_2 = -\frac{a}{2}.$$

- In this case, $z(t) = e^{-\frac{a}{2}t}$ is a solution for the ODE (1).

We show that $z(t) = te^{-\frac{a}{2}t}$ is another solution for (1), by observing that

$$\dot{z}(t) = e^{-\frac{a}{2}t} - \frac{a}{2}te^{-\frac{a}{2}t},$$
$$\ddot{z}(t) = -ae^{-\frac{a}{2}t} + bte^{-\frac{a}{2}t},$$

leading to

$$\ddot{z}(t) + a\dot{z}(t) + bz(t) = -ae^{-\frac{a}{2}t} + bte^{-\frac{a}{2}t} + a\left(e^{-\frac{a}{2}t} - \frac{a}{2}te^{-\frac{a}{2}t}\right) + bte^{-\frac{a}{2}t} = 0.$$

- Therefore, the general solution to (1) is

$$z(t) = c_1 e^{-\frac{at}{2}} + c_2 te^{-\frac{at}{2}},$$

  where c_1, c_2 are real constants determined by the initial values.

Example 2

Example

Consider a second-order ODE

$$\ddot{z}(t) + 6\dot{z}(t) + 9z(t) = 0, \quad z(0) = 2, \quad \dot{z}(0) = -4 \tag{4}$$

- $s^2 + 6s + 9 = (s+3)^2$ has a double root -3.
- The general solution is $z(t) = c_1 e^{-3t} + c_2 t e^{-3t}$, where c_1 and c_2 satisfy

$$\begin{cases} c_1 = 2, \\ -3c_1 + c_2 = -4, \end{cases} \quad \Rightarrow \quad c_1 = 2, c_2 = 2.$$

- Thus, the solution to (4) is $z(t) = 2e^{-3t} + 2te^{-3t}$.
- We can verify this solution by

$$\dot{z}(t) = -4e^{-3t} - 6te^{-3t}, \qquad \ddot{z}(t) = 6e^{-3t} + 18te^{-3t}$$

$$\begin{aligned} & \ddot{z}(t) + 6\dot{z}(t) + 9z(t) \\ \Longrightarrow \quad & = 6e^{-3t} + 18te^{-3t} + 6\left(-4e^{-3t} - 6te^{-3t}\right) + 9\left(2e^{-3t} + 2te^{-3t}\right) \\ & = 0. \end{aligned}$$

Case 3: A pair of complex roots

If $a^2 - 4b < 0$, we have a pair of complex roots

$$s_1 = \frac{-a + i\sqrt{4b - a^2}}{2}, \quad s_2 = \frac{-a - i\sqrt{4b - a^2}}{2}.$$

- The general solution to (1) can be written

$$z(t) = C_1 e^{\frac{1}{2}\left(-a+i\sqrt{4b-a^2}\right)t} + C_2 e^{\frac{1}{2}\left(-a-i\sqrt{4b-a^2}\right)t}$$

- The Euler's identity:

$$e^{it} = \cos(t) + i\sin(t).$$

- Using this identity and substituting

$$C_1 + C_2 = c_1,$$
$$-i(C_1 - C_2) = c_2,$$

  we have the general solution is

$$z(t) = c_1 e^{-\frac{a}{2}t} \cos\left(\frac{1}{2}t\sqrt{4b - a^2}\right) + c_2 e^{-\frac{a}{2}t} \sin\left(\frac{1}{2}t\sqrt{4b - a^2}\right)$$

  where c_1, c_2 are real constants determined by the initial values.

Example 3

Example

Consider a second-order ODE

$$\ddot{z}(t) - 6\dot{z}(t) + 13z(t) = 0, \quad z(0) = 3, \quad \dot{z}(0) = 17 \tag{5}$$

- $s^2 - 6s + 13 = 0$ has a pair of complex roots: $3 + 2i$ and $3 - 2i$.
- Hence, the general solution is

  $$z(t) = c_1 e^{3t} \cos(2t) + c_2 e^{3t} \sin(2t)$$

  where c_1 and c_2 satisfy

  $$\begin{cases} c_1 = 3, \\ 3c_1 + 2c_2 = 17, \end{cases} \implies c_1 = 3, c_2 = 4.$$

- We can verify this solution by observing that

  $$\begin{aligned} &\dot{z}(t) = e^{3t}(17\cos(2t) + 6\sin(2t)), \qquad \ddot{z}(t) = e^{3t}(63\cos(2t) - 16\sin(2t)) \\ \implies \quad &\ddot{z}(t) - 6\dot{z}(t) + 13z(t) = 0. \end{aligned}$$

Outline

Matrix form

As discussed in Lecture 2, any nth linear ODE can be rewritten into

$$\dot{x} = Ax$$

for which we have a general solution $x(t) = e^{At}x(0)$.

- For the second-order ODE in (1), we define

$$x_1(t) = z(t), \qquad x_2(t) = \dot{z}(t).$$

- Then the second-order ODE in (1) becomes

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} x, \qquad \text{with } x(0) = x_0 \in \mathbb{R}^2. \tag{6}$$

Definition

Given a matrix $A \in \mathbb{R}^{n\times n}$, the exponential of A, denoted by e^A, is defined by

$$e^A := I + A + \frac{1}{2}A^2 + \ldots + \frac{1}{n!}A^n + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k.$$

Diagonal matrix

For diagonal matrices, we have

$$\begin{aligned} e^{\Lambda} &= I + \Lambda + \frac{1}{2}\Lambda^2 + \frac{1}{3!}\Lambda^3 + \dots \\ &= \begin{bmatrix} 1 + \lambda_1 + \frac{1}{2}\lambda_1^2 + \frac{1}{3!}\lambda_1^3 + \dots & 0 \\ 0 & 1 + \lambda_2 + \frac{1}{2}\lambda_2^2 + \frac{1}{3!}\lambda_2^3 + \dots \end{bmatrix} \\ &= \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix}, \end{aligned}$$

- Let v_1, v_2 satisfy

$$\begin{cases} Av_1 = \lambda_1 v_1, \\ Av_2 = \lambda_2 v_2 \end{cases} \implies A \underbrace{\begin{bmatrix} v_1 & v_2 \end{bmatrix}}_{P} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{\Lambda}$$

- Thus, we have

$$P^{-1}AP = \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{\Lambda}.$$

Diagonalization

We have

$$\begin{aligned} e^{\Lambda} = e^{P^{-1}AP} &= I + P^{-1}AP + \frac{1}{2}(P^{-1}AP)^2 + \frac{1}{3!}(P^{-1}AP)^3 + \dots \\ &= P^{-1}\left(I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots\right)P \\ &= P^{-1}e^{A}P \end{aligned}$$

This leads to

$$e^{A} = Pe^{\Lambda}P^{-1}$$

Example 4

Example

Consider the ODE (3). It is equivalent to

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 18 & 3 \end{bmatrix} x \qquad \text{with } \; x(0) = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

- In this case, we have

$$A \times \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} = -3 \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}, \qquad A \times \begin{bmatrix} \frac{1}{6} \\ 1 \end{bmatrix} = 6 \begin{bmatrix} \frac{1}{6} \\ 1 \end{bmatrix}$$

- Thus we have

$$P = \begin{bmatrix} -\frac{1}{3} & \frac{1}{6} \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} -2 & \frac{\frac{1}{3}}{\frac{2}{3}} \\ 2 & \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -3 & 0 \\ 0 & 6 \end{bmatrix}.$$

- Finally we have

$$x(t) = e^{At}x(0) = Pe^{\Lambda t}P^{-1}x(0) = \begin{bmatrix} -\frac{1}{3} & \frac{1}{6} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{6t} \end{bmatrix} \begin{bmatrix} -2 & \frac{\frac{1}{3}}{\frac{2}{3}} \\ 2 & \end{bmatrix} \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-3t} + 2e^{6t} \\ -3e^{-3t} + 12e^{6t} \end{bmatrix},$$

Outline

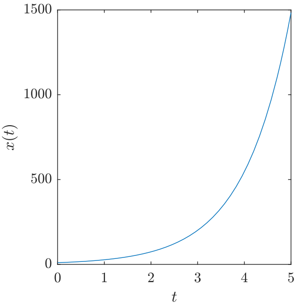
ode45 Matlab

- Matlab ODE45 function:

```
[t,y] = ode45(odefun,tspan,y0)
```

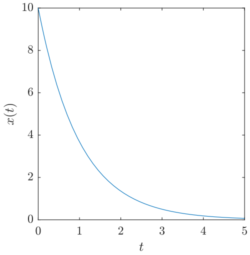
- Many useful information can be found here
  https://www.mathworks.com/help/matlab/ref/ode45.html

```
%----- Example 1 -------
%  \dot x = x,
%   with x(0) = 10
%-----------------------

f1 = @(t,x)(x); % vector field
[ts,ys] = ode45(f1,[0,10],10);
```

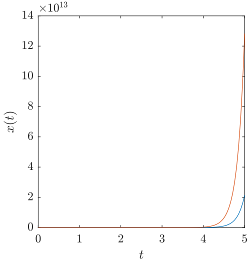


ode45 Matlab - Example 2 & 3

```
%------ Example 2 ------------
%  \dot x = -x,
%   with x(0) = 10
%----------------------------
f2      = @(t,x)(-x); % vector field
[ts,ys] = ode45(f2,[0,5],10);
```

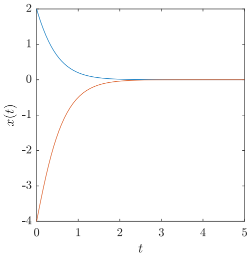



```
%------ Example 3 --------
%  \ddot z - 3 \dot z - 18 z = 0
%   with z(0) = 3, \dot z(0) = 9
%-----------------------
[ts,ys] = ode45(@f3,[0,5],[3;9]);

function dotx = f3(t,x)
   dotx = zeros(2,1);
   dotx(1) = x(2);
   dotx(2) = 18*x(1)+3*x(2);
end
```



ode45 Matlab - Example 4 & 5

```
%------ Example 4 -------------
%  \ddot z + 6 \dot z + 9z = 0
%   with z(0) = 2, \dot z(0) = -4
%-----------------------------
[ts,ys] = ode45(@f4,[0,5],10);

function dotx = f4(t,x)
   dotx = zeros(2,1);
   dotx(1) = x(2);
   dotx(2) = -9*x(1)-6*x(2);
end
```




```
%------ Example 5 --------
%  \ddot z - 6 \dot z + 13 z = 0
%   with z(0) = 3, \dot z(0) = 17
%------------------------
[ts,ys] = ode45(@f5,[0,20],[3;17]);

function dotx = f5(t,x)
   dotx = zeros(2,1);
   dotx(1) = x(2);
   dotx(2) = -13*x(1)+6*x(2);
end
```