ECE 171A: Linear Control System Theory
Discussion 3: Review on Eigenvalues and Eigenvectors

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Outline

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Eigenvalues and Eigenvectors

Diagonalization

Cayley-Hamilton Theorem

Eigenvalues and Eigenvectors

Let $A \in \mathbb{R}^{n\times n}$. If we have

$$Ax = \lambda x, \qquad \lambda \in \mathbb{R}, \qquad x \neq 0 \in \mathbb{R}^n,$$

then λ is called an **eigenvalue** and x is called an **eigenvector** of A.

- **Geometrical interpretation**: if we start along vector x, transforming by A simply scales the vector without affecting its direction.
- How do we find an eigenvalue and eigenvector?
- Observation 1:
$$Ax = \lambda x \Rightarrow (A - \lambda I)x = 0.$$
It means that $A - \lambda I$ is rank deficient ($\mathrm{rank}(A - \lambda I) < n$).
- Then, we have
$$\det(A - \lambda I) = 0.$$
which gives n eigenvalues (multiplicity is counted).

Example 1

Example

Let

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

- **Step 1**: determinant

$$\begin{aligned} \det\left(A - \lambda I\right) &= \det\left(\begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix}\right) \\ &= (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5 = 0 \end{aligned}$$

- **Step 2**: solving the characteristic polynomial

$$\lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0 \Rightarrow \lambda_1 = 5, \lambda_2 = -1.$$

- **Step 3**: find the eigenvector associated with each eigenvalue.

Example 1

Example

- Case 1: $\lambda_1 = 5$

$$(A - \lambda_1 I)x = 0 \Rightarrow \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} -4x_1 + 4x_2 = 0 \\ 2x_1 - 2x_2 = 0. \end{cases} \quad \Rightarrow x_2 = x_1,$$

then $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A associated with $\lambda_1 = 5$.

- Case 2: $\lambda_2 = -1$

$$(A - \lambda_2 I)x = 0 \Rightarrow \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} 2x_1 + 4x_2 = 0 \\ 2x_1 + 4x_2 = 0. \end{cases} \quad \Rightarrow -2x_2 = x_1,$$

then $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector of A associated with $\lambda_2 = -1$.

Example 2

Example

Consider a second-order ODE

$$\ddot{z}(t) - 3\dot{z}(t) - 18z(t) = 0, \quad z(0) = 3, \quad \dot{z}(0) = 9 \tag{1}$$

- The characteristic polynomial is $s^2 - 3s - 18 = (s-6)(s+3)$ which has roots 6 and -3.
- It is equivalent to

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 18 & 3 \end{bmatrix} x \qquad \text{with } \; x(0) = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

- In this case, we have the eigenvalues and eigenvectors of A as

$$A \times \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} = -3 \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}, \qquad A \times \begin{bmatrix} \frac{1}{6} \\ 1 \end{bmatrix} = 6 \begin{bmatrix} \frac{1}{6} \\ 1 \end{bmatrix}$$

Examples in Matlab

Matlab eig function:

```
[V,D] = eig(A)
```

- It returns diagonal matrix D of eigenvalues and matrix V whose columns are the corresponding right eigenvectors, so that $A \times V = V \times D$.
- Useful information can be found here `https://www.mathworks.com/help/matlab/ref/eig.html`

What are the eigenvalues and eigenvectors of

$$A_2 = \begin{bmatrix} 12 & 3 \\ 2 & 7 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & 5 & 4 \\ 2 & 5 & 1 \\ 7 & 4 & 1 \end{bmatrix},$$

2nd ODE and its matrix form

A **homogeneous second-order linear ODEs** (with constant coefficients) is in the form

$$\ddot{z}(t) + a\dot{z}(t) + bz(t) = 0 \qquad (2)$$

where $a \in \mathbb{R}$ and $b \in \mathbb{R}$ are constants.

- Let us define

$$x_1 = z, \qquad x_2 = \dot{z}.$$

- Then (2) is equivalent to a first-order **matrix ODE**

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- What are the eigenvalues of A?

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ -b & -a-\lambda \end{bmatrix}\right) = \lambda^2 + a\lambda + b = 0$$

which is identical to the **characteristic polynomial** of (2).

Review on solving 2nd ODE

We guess the solution to (2) is in the form of $z(t) = e^{st}$.

- Substituting the (2) by $z(t) = e^{st}$, we obtain that
$$\ddot{z}(t) + a\dot{z}(t) + bz(t) = s^2 e^{st} + ase^{st} + be^{st} = e^{st}(s^2 + as + b) = 0$$
- The value of s must satisfy
$$F(s) := s^2 + as + b = 0. \tag{3}$$
- $F(s)$ is called the **characteristic polynomial** associated with a homogeneous second-order ODE.

Solving the original ODE is reduced to solving an algebraic equation.
Three cases:

1. $F(s)$ has two distinct roots;
2. $F(s)$ has a double root;
3. $F(s)$ has a pair of complex roots;

Why do we care?

In control theory, we can use the eigenvalues of a system to make a statement about its stability.

- The precise definition of stability is discussed in Lecture 7.

Theorem (Stability of a linear system)

The system

$$\dot{x} = Ax$$

- *is* **asymptotically stable** *if and only if all eigenvalues of A have a strictly negative real part, i.e.,* $\mathrm{Re}(\lambda_i) < 0$
- *is* **unstable** *if any eigenvalues A has a strictly positive real part.*

Remark: This result works for any LTI system (beyond 2nd ODE). If $\mathrm{Re}(\lambda_i) \leq 0, i = 1, \ldots, n$ and some $\mathrm{Re}(\lambda_i) = 0$, the stability conditions are more complicated, which is beyond the scope of this class.

Unstable systems

Example (Unstable systems)

Consider the system $\ddot{q} = 0$. It can be written in state-space form as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

- The system has eigenvalues $\lambda = 0$, but the solutions are not bounded

$$\begin{aligned} x_1(t) &= x_1(0) + x_2(0)t, \\ x_2(t) &= x_2(0). \end{aligned}$$

Outline

Matrix form

As discussed in Lecture 2, any nth linear ODE can be rewritten into

$$\dot{x} = Ax$$

for which we have a general solution $x(t) = e^{At}x(0)$.

- For the second-order ODE in (2), we define

$$x_1(t) = z(t), \qquad x_2(t) = \dot{z}(t).$$

- Then the second-order ODE in (2) becomes

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} x, \qquad \text{with } x(0) = x_0 \in \mathbb{R}^2. \tag{4}$$

Definition

Given a matrix $A \in \mathbb{R}^{n \times n}$, the exponential of A, denoted by e^A, is defined by

$$e^A := I + A + \frac{1}{2}A^2 + \ldots + \frac{1}{n!}A^n + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k.$$

Diagonal matrix

For a diagonal matrix Λ, we have

$$\begin{aligned} e^{\Lambda} &= I + \Lambda + \frac{1}{2}\Lambda^2 + \frac{1}{3!}\Lambda^3 + \dots \\ &= \begin{bmatrix} 1 + \lambda_1 + \frac{1}{2}\lambda_1^2 + \frac{1}{3!}\lambda_1^3 + \dots & 0 \\ 0 & 1 + \lambda_2 + \frac{1}{2}\lambda_2^2 + \frac{1}{3!}\lambda_2^3 + \dots \end{bmatrix} \\ &= \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix}, \end{aligned}$$

Diagonalization

- For a diagonalizable matrix A, let v_1, v_2 satisfy

$$\begin{cases} Av_1 = \lambda_1 v_1, \\ Av_2 = \lambda_2 v_2 \end{cases} \implies A \underbrace{\begin{bmatrix} v_1 & v_2 \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} v_1 & v_2 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{\Lambda}$$

- Thus, we have

$$A = P\Lambda P^{-1}.$$

We have

$$\begin{aligned} e^A = e^{P\Lambda P^{-1}} &= I + P\Lambda P^{-1} + \frac{1}{2}(P\Lambda P^{-1})^2 + \frac{1}{3!}(P\Lambda P^{-1})^3 + \dots \\ &= P\left(I + \Lambda + \frac{1}{2}\Lambda^2 + \frac{1}{3!}\Lambda^3 + \dots\right)P^{-1} \\ &= Pe^{\Lambda}P^{-1} \end{aligned}$$

Diagonalization

We can use eigenvectors and eigenvalues to diagonalize A in special cases

In this class, we will deal with diagonalizable matrices often. Some examples of diagonalizable matrices are

- Symmetric matrices;
- All eigenvalues are distinct;
- The matrix A has n linearly independent eigenvectors.

$$e^{At} = P \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix} P^{-1}$$

If there is an eigenvalue with $\mathrm{Re}(\lambda_i) > 0$, the system state will be growing unboundedly along that eigenvector.

Outline

Cayley-Hamilton Theorem

Theorem

Let $\lambda^n + a_{n-1}\lambda^{n-1} + \dots a_1\lambda + a_0 = 0$ *be the characteristic equation of* A*, i.e.,* $\det(\lambda I - A) = 0$*. Then, we have*

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0$$

Some implications:

- Cayley-Hamilton Theorem says that $I, A, A^2, A^3, \dots, A^n$ are linearly dependent.
- It also shows that the inverse of A is a linear combination of its power sequences up to A^{n-1}

$$A^{-1} = -\frac{1}{a_0}(A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I).$$