ECE 171A: Linear Control System Theory
Discussion 6: Bode plot - Examples

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Outline

First-order and second-order examples

High-order examples

The Routh–Hurwitz Criterion: Examples

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The Routh–Hurwitz Criterion: Examples

Bode plot

The **frequency response** of a **stable** linear system can be computed from its transfer function by setting $s = i\omega$, i.e.,

$$u(t) = e^{i\omega t} = \cos(\omega t) + i\sin(\omega t).$$

- The resulting steady-state output is

$$y(t) = G(i\omega)e^{i\omega t} = Me^{i(\omega t+\theta)} = M\cos(\omega t + \theta) + iM\sin(\omega t + \theta)$$

- Thus, we have $\cos(\omega t) \to M\cos(\omega t + \theta)$ and $\sin(\omega t) \to M\sin(\omega t + \theta)$

The frequency response $G(i\omega)$ can be represented by two curves — **Bode plot**

- **Gain curve**: gives $|G(i\omega)|$ as a function of frequency ω — log/log scale (traditionally often in dB — $20\log|G(i\omega)|$; but we mainly use $\log|G(i\omega)|$)
- **Phase curve**: gives $\angle G(i\omega)$ as a function of frequency ω — log/linear scale in degrees

Case 2: first-order system

Consider the transfer function of a first-order system

$$G(s) = \frac{a}{s+a}, \qquad a > 0.$$

- We have

$$|G(s)| = \frac{|a|}{|s+a|}, \qquad \angle G(s) = \angle a - \angle(s+a).$$

- The gain curve is

$$\log|G(i\omega)| = \log a - \frac{1}{2}\log(\omega^2 + a^2) \approx \begin{cases} 0, & \text{if } \omega < a \\ \log a - \log\omega, & \text{if } \omega > a \end{cases}$$

- The phase curve is

$$\angle G(i\omega) = -\frac{180}{\pi}\arctan\frac{\omega}{a} \approx \begin{cases} 0, & \text{if } \omega < \frac{a}{10} \\ -45 - 45(\log\omega - \log a), & \text{if } a/10 < \omega < 10a \\ -90, & \text{if } \omega > 10a \end{cases}$$

Case 2: first-order system

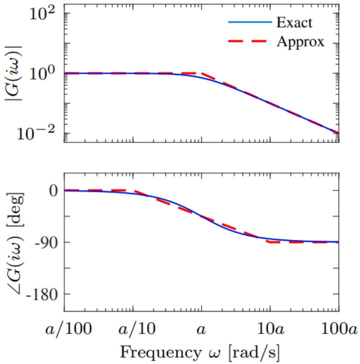


Figure: Bode plot of the first-order system $G(s) = a/(s+a)$, which can be approximated by asymptotic curves (dashed) in both the gain and the frequency, with the breakpoint in the gain curve at $\omega = a$ and the phase decreasing by 90° over a factor of 100 in frequency.

A first-order system behaves like a constant for low frequencies and like an integrator for high frequencies.

Case 3: second-order system

Consider the transfer function of a second-order system

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}, \qquad 0 < \zeta < 1.$$

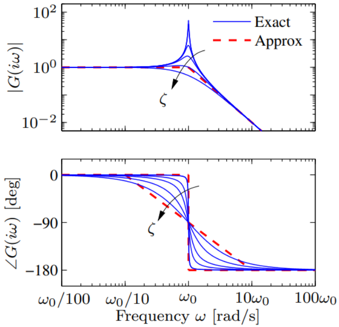
- The gain curve is

$$\begin{aligned} \log|G(i\omega)| &= 2\log\omega_0 - \frac{1}{2}\log(\omega^4 + 2\omega_0^2\omega^2(2\zeta^2 - 1) + \omega_0^4) \\ &\approx \begin{cases} 0, & \text{if } \omega \ll \omega_0 \\ 2\log\omega_0 - 2\log\omega, & \text{if } \omega \gg \omega_0 \end{cases} \end{aligned}$$

- The largest gain $Q = \max_\omega |G(i\omega)| \approx 1/(2\zeta)$, called the Q-value, is obtained for $\omega \approx \omega_0$ – **Resonant frequency**
- The phase curve is

$$\angle G(i\omega) = -\frac{180}{\pi}\arctan\frac{2\zeta\omega_0\omega}{\omega_0^2 - \omega^2} \approx \begin{cases} 0, & \text{if } \omega \ll \omega_0 \\ -180, & \text{if } \omega \gg \omega_0 \end{cases}$$

Case 3: Second-order system



(b) Second-order system

Figure: Bode plot of the second-order system $G(s) = \omega_0^2/(s^2 + 2\zeta\omega_0 s + \omega_0^2)$, which has a peak at frequency ω_0 and then a slope of -2 beyond the peak; the phase decreases from 0° to -180°. The height of the peak and the rate of change of phase depending on the damping ratio ζ ($\zeta = 0.02, 0.1, 0.2, 0.5$, and 1.0 shown).

The asymptotic approximation is poor near $\omega = \omega_0$ and that the Bode plot depends strongly on ζ near this frequency.

Determine Transfer function experimentally

Model a given application by measuring the frequency response

- Apply a sinusoidal signal at a fixed frequency.
- Measure the amplitude ratio and phase lag when steady state is reached.
- The complete frequency response is obtained by sweeping over a range of frequencies.

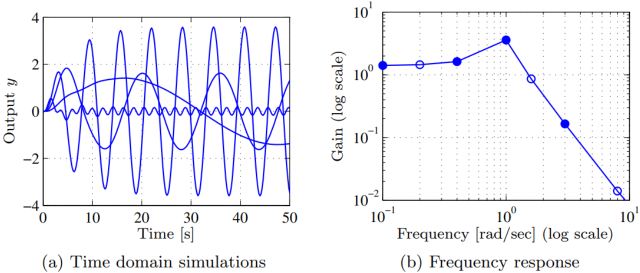


Figure: A frequency response (gain only) computed by measuring the response of individual sinusoids.

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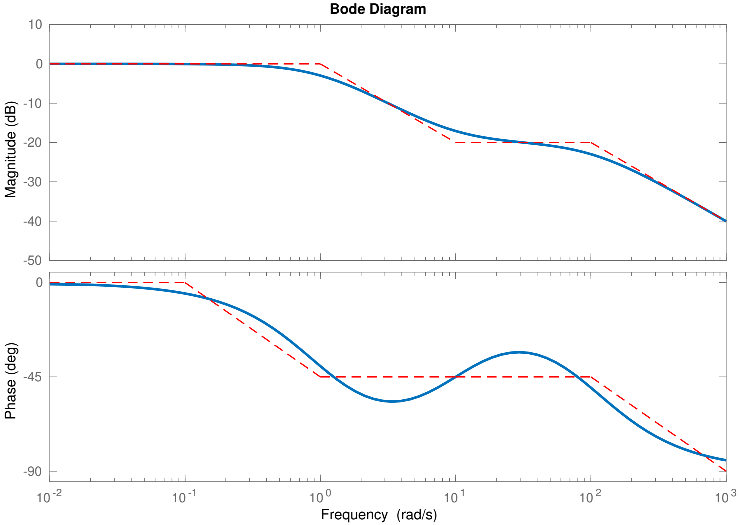
High-order Example I

Example

Draw a Bode plot for $G(s) = 10\frac{s+10}{(s+1)(s+100)}$

- Step 1: find break points (related to poles and zeros): 1, 10, 100.
- Step 2: Calculate $|G(i0)|$ and $\angle G(i0)$ to determine the starting points
- Step 3: Sketch the bode plot by the rules
  - **Magnitude increases with a zero**: if the zero is a first-order real zero, the slope is $+1$; if the zero is a second-order zero (or complex zero), the slope is $+2$
  - **Magnitude decreases with a pole**: If the pole is a first-order real pole, the slope is -1; if the pole is a second-order pole (or complex pole), the slope is -2
  - **Phases changes** by $+90$ with a first-order real zero; $+180$ with a second-order zero (or complex zero). The change starts around $a/10$ and ends around $10a$.
  - **Phases changes** by -90 with a first-order real pole; -180 with a second-order pole (or complex pole). Similarly, the change starts around $a/10$ and ends around $10a$.

High-order Example I



Another example II

Example

Draw bode plot for

$$G(s) = \frac{k(s+b)}{(s+a)(s^2+2\zeta\omega_0 s+\omega_0^2)}, \qquad a \ll b \ll \omega_0.$$

- **Gain curve:**
  - Begin with low frequency $G(0) = \frac{kb}{a\omega_0^2}$.
  - Reach $\omega = a$, the effect of the pole begins and the gain decreases with slope -1
  - At $\omega = b$, the zero comes into play and we increase the slope by 1, leaving the asymptote with net slope 0.
  - This slope is used until the effect of the second-order pole is seen at $\omega = \omega_0$, at which point the asymptote changes to slope -2.
- **Phase curve:**
  - The approximation process is similar, but slightly more complicated since the effect of the phase stretches out much further.

Another example II

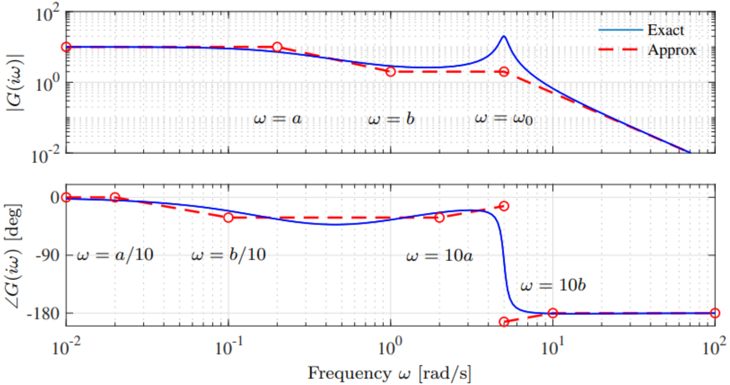


Figure 9.15: Asymptotic approximation to a Bode plot. The solid curve is the Bode plot for the transfer function $G(s) = k(s+b)/(s+a)(s^2+2\zeta\omega_0 s+\omega_0^2)$, where $a \ll b \ll \omega_0$. Each segment in the gain and phase curves represents a separate portion of the approximation, where either a pole or a zero begins to have effect. Each segment of the approximation is a straight line between these points at a slope given by the rules for computing the effects of poles and zeros.

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Stability

Theorem (Stability of a linear system (Lyapunov sense))

The system $\dot{x} = Ax$ is

- **asymptotically stable** *if and only if all eigenvalues of A have a strictly negative real part, i.e.,* $\mathrm{Re}(\lambda_i) < 0$
- **unstable** *if any eigenvalues A has a strictly positive real part.*

Consider an LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du \end{aligned} \qquad \Longleftrightarrow \qquad G(s) = C(sI - A)^{-1}B + D$$

Poles (eigenvalues) of the matrix A = Poles of the transfer function $G(s)$

- A system is **bounded-input bounded-output (BIBO)** stable if every bounded input $u(t)$ leads to a bounded output $y(t)$.
- BIBO stable: if all poles of $G(s)$ are in the open left half-plane in the s domain (i.e., having negative real parts).

Routh Table

- $a(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0$

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$	a_0
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$	0
s^{n-2}	$b_{n-1} = -\frac{\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}}$	$b_{n-3} = -\frac{\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{a_{n-1}}$	b_{n-5}	$\cdots$	0
s^{n-3}	$c_{n-1} = -\frac{\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}}{b_{n-1}}$	$c_{n-3} = -\frac{\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_{n-1} & b_{n-5} \end{vmatrix}}{b_{n-1}}$	c_{n-5}	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$
s^0	a_0	0	0	$\cdots$	0

- Any row can be multiplied by a positive constant without changing the result

Example: Higher-order System

Example

Consider the characteristic polynomial of a fifth-order system:

$$a(s) = s^5 + s^4 + 10s^3 + 72s^2 + 152s + 240$$

- The Routh table is:

s^5	1	10	152
s^4	1	72	240
s^3	-62	-88	0
s^2	70.6	240	0
s^1	122.6	0	0
s^0	240	0	0

- Since there are two sign changes in the first column, there are two roots in the right half-plane and the system is **unstable**
- The roots of $a(s)$ are:

$$a(s) = (s+3)(s+1 \pm j\sqrt{3})(s-2 \pm j4)$$

Example: Special Case

Example

Consider the polynomial:

$$a(s) = s^4 + s^3 + 2s^2 + 2s + 3$$

- The Routh table is:

s^4	1	2	3
s^3	1	2	0
s^2	$\not{0} \to \epsilon$	3	0
s^1	$2 - \frac{3}{\epsilon}$	0	0
s^0	3	0	0

- For $0 < \epsilon \ll 1$, we see that $2 - \frac{3}{\epsilon} < 0$
- Since there are two sign changes in the first column, there are two roots in the right half-plane and the system is **unstable**
- The roots are $p_{1,2} = 0.4057 \pm 1.2928i, \qquad p_{3,4} = -0.9057 \pm 0.9020i$

Example: Special Case 2

Example

Consider the polynomial:

$$a(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

- The Routh table is:

s^5	1	2	11
s^4	2	4	10
s^3	$\cancel{0} \rightarrow \epsilon$	6	0
s^2	$c_4 = \frac{1}{\epsilon}(4\epsilon - 12)$	10	0
s^1	$d_4 = \frac{1}{c_4}(6c_4 - 10\epsilon)$	0	0
s^0	10	0	0

- For $0 < \epsilon \ll 1$, we see that $c_4 < 0$ and $d_4 > 0$
- Since there are two sign changes in the first column, there are two roots in the right half-plane and the system is **unstable**
- The roots are
$\lambda_{1,2} = 0.8950 \pm 1.4561i, \lambda_{3,4} = -1.2407 \pm 1.0375i, \lambda_5 = -1.3087.$