ECE 171A: Linear Control System Theory
Discussion 7: Nyquist plot - Review & Examples

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Outline

Nyquist stability criterion

Nyquist plot - examples

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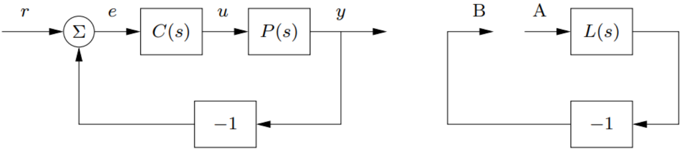
Stability of feedback systems



(a) Closed loop system (b) Open loop system

- **Lyapunov stability** — eigenvalue test of the closed-loop matrix; e.g.,

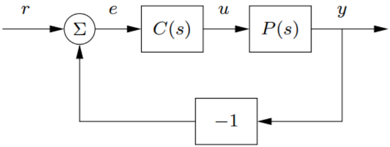
$$\begin{aligned} \text{Dynamics} \to \quad & \dot{x} = Ax + Bu, \\ \text{Feedback controller} \to \quad & u = -Kx \end{aligned} \quad \Rightarrow \quad \dot{x} = (A - BK)x.$$

- **Poles or The Routh–Hurwitz Criterion**;

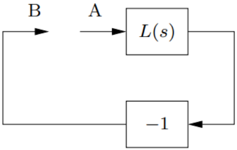
$$\begin{cases} P(s) &= \dfrac{n_{\mathrm{p}}(s)}{d_{\mathrm{p}}(s)} \\ C(s) &= \dfrac{n_{\mathrm{c}}(s)}{d_{\mathrm{c}}(s)} \end{cases} \quad \Rightarrow \quad G_{yr}(s) = \frac{PC}{1 + PC} = \frac{n_{\mathrm{p}}(s)n_{\mathrm{c}}(s)}{d_{\mathrm{p}}(s)d_{\mathrm{c}}(s) + n_{\mathrm{p}}(s)n_{\mathrm{c}}(s)}$$

They are **straightforward but give little guidance** for design: it is not easy to tell how the controller should be modified to make an unstable system stable.

Nyquist's idea



(a) Closed loop system



(b) Open loop system

- Nyquist's idea was to first investigate conditions under which oscillations can occur in a feedback loop.
- The **Loop transfer function**:

$$L(s) = P(s)C(s).$$

- Assume that a sinusoid of frequency ω_0 is injected at point A. In steady state, the signal at point B will also be a sinusoid with the frequency ω_0.

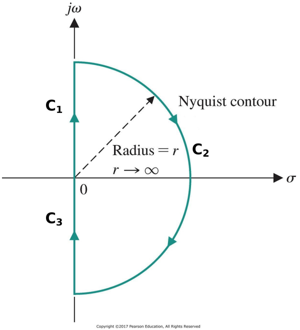
Very intuitive idea: It seems reasonable that an oscillation can be maintained if the signal at B has the same amplitude and phase as the injected signal!

- In this case, $L(i\omega_0) = -1$ (thus, $s = -1 + i0$) is called the critical point).

Nyquist contour

The (standard or simplest) Nyquist contour, also known as "Nyquist D contour" ($\Gamma \subset \mathbb{C}$), is made up of three parts:

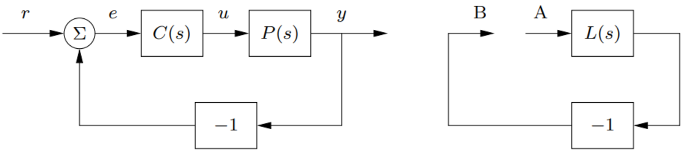
- **Contour** C_1: points $s = i\omega$ on the positive imaginary axis, as ω ranges from 0 to ∞
- **Contour** C_2: points $s = Re^{i\theta}$ on a semi-circle as $R \to \infty$ and θ ranges from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$
- **Contour** C_3: points $s = i\omega$ on the negative imaginary axis, as ω ranges from $-\infty$ to 0



The image of $L(s)$ when s traverses Γ gives a closed curve in the complex plane and is referred to as the **Nyquist plot** for $L(s)$.

- **Nyquist's stability criterion** utilizes **contours** in the complex plane to relate the **locations** of the open-loop and closed-loop poles.

Simplified Nyquist Criterion



(a) Closed loop system (b) Open loop system

Theorem (Simplified Nyquist Criterion)

Let $L(s)$ be the loop transfer function for a negative feedback system, and assume that L has no poles in the closed right half-plane ($\mathrm{Re}(s) \geq 0$) except possibly at the origin. Then the closed loop system

$$G_{\mathrm{cl}}(s) = \frac{L(s)}{1 + L(s)}$$

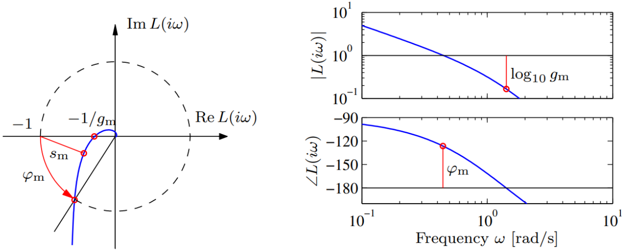
is stable if and only if the image of $L(s)$ along the closed contour Γ (i.e., its Nyquist plot) has no net encirclements of the critical point $s = -1$.

Nyquist's Stability Criterion

Theorem (Nyquist Stability Criterion)

Consider a unity feedback control system with open-loop transfer function $L(s)$. Let Γ be a Nyquist contour. The closed-loop system is stable if and only if **the number of counterclockwise encirclements** *of the critical point $-1 + i0$ by the Nyquist plot $L(\Gamma)$ is equal to* **the number of open-loop unstable poles** *of $L(s)$.*

Classical robustness measures: stability margin, phase margin, gain margin



(a) Nyquist plot (b) Bode plot

Outline

Nyquist stability criterion

Nyquist plot - examples

Example 1: a third-order system

Draw a Nyquist plot for $L(s) = \frac{1}{(s+a)^3}$.

- **Counter** C_1: $s = i\omega$ with ω from 0 to ∞

$$L(i0) = \frac{1}{a^3}\angle 0^\circ, \qquad L(i\infty) = 0\angle -270^\circ$$

- for $0 < \omega < \infty$

$$L(i\omega) = \frac{1}{(i\omega + a)^3}$$

- **Counter** C_2: $s = Re^{i\theta}$ for $R \to \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$.

$$L(Re^{i\theta}) = \frac{1}{(Re^{i\theta} + a)^3} \to 0$$

- **Counter** C_3: $s = i\omega$ with $\omega \in (-\infty, 0)$

$$L(-i\omega) = L(\bar{i}\omega) = \overline{L(i\omega)}$$

which is a *reflection* (complex conjugate) of $L(C_1)$ about the real axis.

Example 1: a third-order system

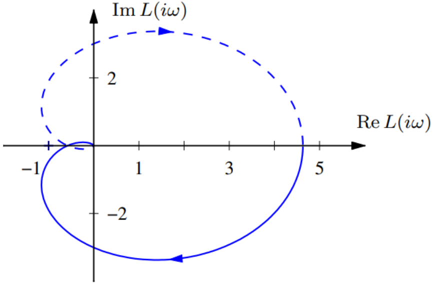


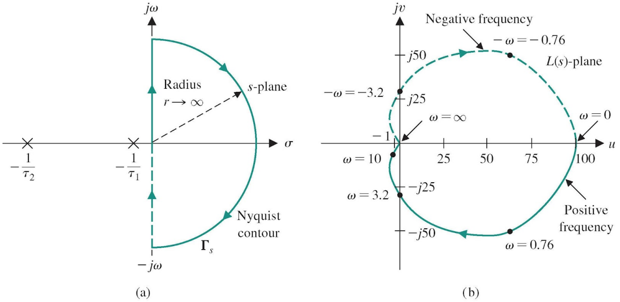
Figure 10.5: Nyquist plot for a third-order transfer function $L(s)$. The Nyquist plot consists of a trace of the loop transfer function $L(s) = 1/(s+a)^3$ with $a = 0.6$. The solid line represents the portion of the transfer function along the positive imaginary axis, and the dashed line the negative imaginary axis. The outer arc of the Nyquist contour Γ maps to the origin.

Example 2: a second-order system

Draw a Nyquist plot for

$$L(s) = \frac{100}{(1+s)(1+s/10)}.$$

- **Contour** C_1: $L(i0) = 100\angle 0°$, $L(i\infty) = 0\angle{-180°}$
- **Contour** C_2: $\lim_{R\to\infty} L(Re^{i\theta}) = 0$



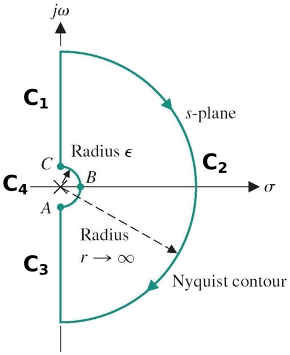
Pole/Zero on the Imaginary Axis

- When the loop transfer function has poles on the imaginary axis, the gain is infinite at the poles.
- The Nyquist contour needs to be modified to take a small detour around such poles or zeros
- So, we add another part: **Contour** C_4
  - plot $L(\epsilon e^{i\theta})$ for $\epsilon \to 0$ and

$$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

  - substitute $s = \epsilon e^{i\theta}$ into $L(s)$ and examine what happens as

$$\epsilon \to 0$$

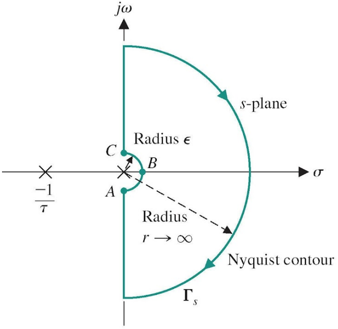


Example 3

Draw a Nyquist plot for a loop transfer system:

$$L(s) = \frac{\kappa}{s(1+\tau s)}$$

- Since there is a pole at the origin, we need to use a modified Nyquist contour



(a) (b)

Example 3

- **Contour** C_4 with $s = \epsilon e^{i\theta}$ for $\epsilon \to 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$:

$$\lim_{\epsilon \to 0} L(\epsilon e^{i\theta}) = \lim_{\epsilon \to 0} \frac{\kappa}{\epsilon e^{i\theta}} = \lim_{\epsilon \to 0} \frac{\kappa}{\epsilon} e^{-i\theta} = \infty \angle -\theta$$

  - The phase of $L(s)$ changes from $\dfrac{\pi}{2}$ at $\omega = 0^-$ to $-\dfrac{\pi}{2}$ at $\omega = 0^+$
- **Contour** C_1 with $\omega \in (0, \infty)$:

$$\begin{aligned} L(i0^+) &= \infty \angle -90^\circ \\ L(i\infty) &= \lim_{\omega \to \infty} \frac{\kappa}{i\omega(1 + i\omega\tau)} \\ &= 0 \angle -180^\circ \end{aligned}$$

- **Contour** C_2 with $s = re^{i\theta}$ for $r \to \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$:

$$\lim_{r \to \infty} L(re^{i\theta}) = \lim_{r \to \infty} \left| \frac{\kappa}{\tau r^2} \right| e^{-2i\theta} = 0 \angle -2\theta$$

  - The phase of $L(s)$ changes from $-\pi$ at $\omega = \infty$ to π at $\omega = -\infty$
- **Contour** C_3 with $\omega \in (-\infty, 0)$:
  - $L(C_3)$ is a **reflection** of $L(C_1)$ about the real axis

Summary - Nyquist contour

- *Open-loop transfer function*: $L(s) = P(s)C(s)$
- *Close-loop transfer function*

$$G_{\mathrm{yr}} = \frac{L(s)}{1 + L(s)}$$

Nyquist Contour is a D-shape curve in the complex domain, avoiding all the poles of $L(s)$ on the imaginary axis.

- Only poles on the imaginary axis needs to be avoided.
- The default orientation of traveling along the contour is clockwise.
- The semi-circle centered at pole p on the imaginary axis, rotating in counter clockwise direction, is represented by $p + Re^{i\theta},\ \theta \sim -\frac{\pi}{2} \to \frac{\pi}{2}$.
- The big semi-circle of the contour, rotating in clockwise direction, is represented by $Re^{i\theta},\ \theta : +\frac{\pi}{2} \to -\frac{\pi}{2}, R \to \infty$

Summary - Nyquist plots

The **Nyquist Plot** is the image of the **Nyquist Contour** after going through the function $L(s)$. Nyquist Contour $\Gamma \Rightarrow$ Nyquist Plot $L(\Gamma)$.

- Start with the expression of $L(s)$ when s is on the imaginary axis $s = i\omega$.
- When drawing the plot, it is helpful to first think about how $|L(s)|$ will change, then think about how $\angle(L(s))$ will change.
- In many cases, $|L(s)| \to 0$ when $|s| \to \infty$. Many Nyquist plots are stuck at 0 as you travel along the big semi-circle of the Nyquist Contour.
- The part of the Nyquist Plot corresponding to the negative imaginary axis in the Nyquist Contour is **symmetrical** (reflection) to the other half.
- Cautions with using MATLAB
  - MATLAB doesn't generate the portion of plot for corresponding to the poles on imaginary axis
  - These must be drawn in by hand (get the orientation right!)

Theorem (Nyquist stability theorem)

$1 + L(s)$ *has* $Z = N + P$ *zeros in the right half plane (i.e.,* **closed-loop unstable poles***), where* P *is the number of open-loop unstable poles and* N *is the number of clockwise encirclements of* -1 *by the Nyquist plot.*

Example 4

$$L(s) = \frac{1}{s+1}$$

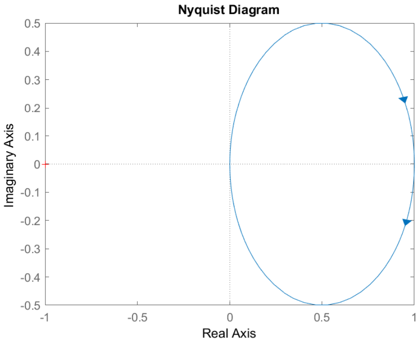


Figure: Nyquist plot for $L(s) = \frac{1}{s+1}$

$$Z = N + P = 0$$

Then,

$$G_{\text{yr}} = \frac{L(s)}{1+L(s)} = \frac{1}{s+2}$$

is stable

Example 5

$$L(s) = \frac{1}{(s+1)^2}$$

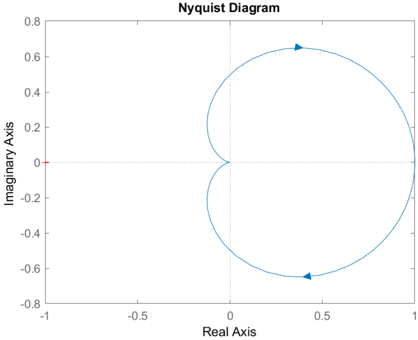


Figure: Nyquist plot for $L(s) = \frac{1}{(s+1)^2}$

$$Z = N + P = 0$$

Then,

$$G_{\text{yr}} = \frac{L(s)}{1 + L(s)} = \frac{1}{s^2 + 2s + 2}$$

is stable. Indeed, closed-loop poles are

$$p_{1,2} = -1 \pm 1i$$

Example 6

$$L(s) = \frac{1}{s(s+1)}$$

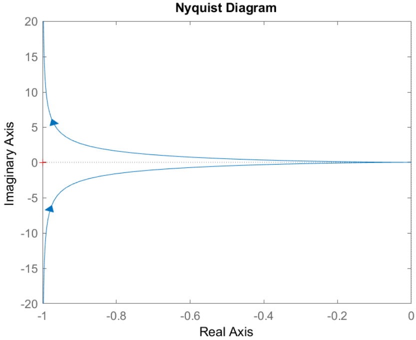


Figure: Nyquist plot for $L(s) = \frac{1}{s(s+1)}$

$$Z = N + P = 0$$

Then,

$$G_{\text{yr}} = \frac{L(s)}{1 + L(s)} = \frac{1}{s^2 + s + 1}$$

is stable. Closed-loop poles

$$p_{1,2} = -0.5 \pm 0.866i$$

Example 7

$$L(s) = \frac{1}{s(s+1)(s+0.5)}$$

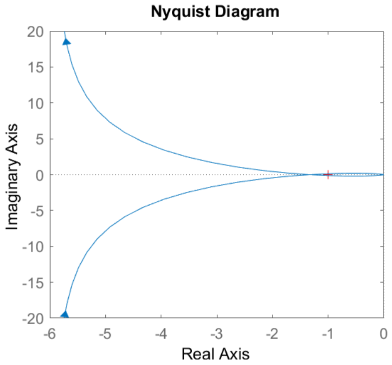


Figure: Nyquist plot for $L(s) = \frac{1}{s(s+1)(s+0.5)}$

$$Z = N + P = 2$$

Then,

$$G_{\text{yr}} = \frac{L(s)}{1+L(s)} = \frac{1}{s^3 + 1.5s^2 + 0.5s + 1}$$

is unstable. Closed-loop poles

$$p_{1,2} = 0.0416 \pm 0.7937i$$
$$p_3 = -1.5832$$