ECE 171A: Linear Control System Theory
Lecture 12: Transfer functions (I)

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Reading materials: Ch 6.3, Ch 9.1, 9.2

Outline

The convolution equation

Consider a state-space system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$.

Theorem

The solution to the linear differential equation (1) *is given by*

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t). \tag{2}$$

Some observations:

- Control design is to design an input signal $u(t)$ to shape $y(t)$ (stability, tracking performance, less overshot, less oscillation, robustness, etc.);
- The solution (2) is too complex to use for designing a controller;
- We look for some elegant and simple tools: make the mapping from $u(t)$ to $y(t)$ easier to compute — **Transfer functions**

Steady-state response

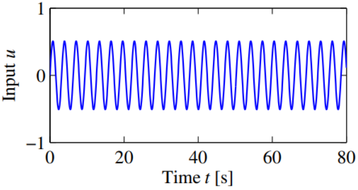
A common practice in evaluating the response of a linear system is to separate out the **short-term response** from the **long-term response**.

- **Transient response:** which occurs in the first period of time after the input is applied.

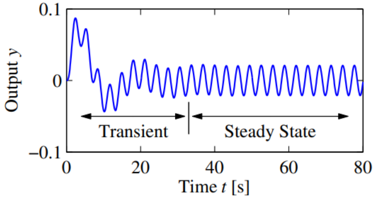
  It reflects the mismatch between the initial condition and the steady-state solution

- **Steady-state response:** which is the portion of the output response that reflects the long-term behavior of the system under the given inputs.
  - For inputs that are periodic, the steady-state response will often be periodic (e.g., **frequency response**)
  - For constant inputs, the response will often be constant (e.g., **step response**)

Example



(a) Input



(b) Output

Figure 6.8: Transient versus steady-state response. The input to a linear system is shown in (a), and the corresponding output with $x(0) = 0$ is shown in (b). The output signal initially undergoes a transient before settling into its steady-state behavior.

Step response

Step input (also known as Heaviside step function)

$$u(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t > 0 \end{cases}$$

- Let's assume $x(0) = 0$
- Solve the step response

$$\begin{aligned} y(t) &= Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \\ &= \int_0^t Ce^{At} \times e^{-A\tau} B d\tau + D = Ce^{At} \int_0^t e^{-A\tau} d\tau B + D \\ &= Ce^{At} \left(-A^{-1} e^{-A\tau} \Big|_{\tau=0}^{\tau=t} \right) B + D \\ &= \underbrace{CA^{-1}e^{At}B}_{\text{transient}} + \underbrace{D - CA^{-1}B}_{\text{steady-state}} \end{aligned}$$

Outline

Frequency response

Determining transfer functions

Block Diagrams and Transfer Functions

Summary

Frequency response

The **frequency response** of an input/output system measures the way in which the system responds to a sinusoidal excitation.

- The particular solution associated with a sinusoidal excitation is itself a sinusoid at the same frequency.
- We can compare the **magnitude** and **phase** of the output sinusoid to the input (— **Transfer function** and **Bode plot**).

Let us consider a sinusoid input

$$u(t) = \cos \omega t.$$

- Evaluating the convolution equation (2) with input $u(t) = \cos \omega t$ can be very messy.
- We use the fact that the system is linear to simplify the derivation.
- In particular, Euler's formula tells us that

$$\cos \omega t = \frac{1}{2} \left(e^{i\omega t} + e^{-i\omega t} \right)$$

- Thanks to the linearity, we can use the exponential input $u(t) = e^{st}$, and then construct the solution by letting $s = i\omega$ and $s = -i\omega$.

Frequency response - derivation

- We apply the convolution equation to $u = e^{st}$

$$
\begin{aligned}
y(t) &= Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Be^{s\tau}d\tau + De^{st} \\
&= Ce^{At}x(0) + Ce^{At}\int_0^t e^{(sI-A)\tau}d\tau B + De^{st}
\end{aligned}
$$

- We assume $(sI - A)$ is invertible, then

$$
\begin{aligned}
y(t) &= Ce^{At}x(0) + Ce^{At}\left((sI-A)^{-1}e^{(sI-A)\tau}\Big|_{\tau=0}^{\tau=t} \right) B + De^{st} \\
&= Ce^{At}x(0) + Ce^{At}(sI-A)^{-1}\left(e^{(sI-A)t} - I \right) B + De^{st} \\
&= Ce^{At}x(0) + C(sI-A)^{-1}e^{st}B - Ce^{At}(sI-A)^{-1}B + De^{st}
\end{aligned}
$$

- Finally, we obtain

$$
y(t) = \underbrace{Ce^{At}\left(x(0) - (sI-A)^{-1}B\right)}_{\text{transient}} + \underbrace{\left(C(sI-A)^{-1}B + D\right)e^{st}}_{\text{steady-state}}
$$

Frequency response - steady-state component

The solution corresponding to the exponential input e^{st} is

$$y(t) = \underbrace{Ce^{At}\left(x(0) - (sI - A)^{-1}B\right)}_{\text{transient}} + \underbrace{\left(C(sI - A)^{-1}B + D\right)e^{st}}_{\text{steady-state}}$$

- If the system matrix A is stable, the transient component decays to zero
- **The steady-state component** is proportional to the exponential input e^{st}.
- We can write the steady-state response as

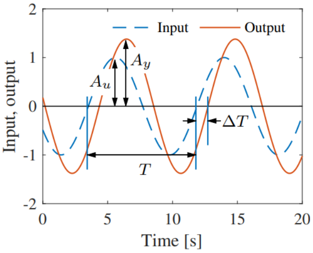
$$y_{\text{ss}}(t) = Me^{i\theta}e^{st} = Me^{st+i\theta},$$

  where $G(s) = C(sI - A)^{-1}B + D \quad \leftarrow$ **Transfer function**
- When $s = i\omega$, we call
  - $M = |G(i\omega)|$ the **gain**, and
  - $\theta = \arg(G(i\omega))$ the **phase** of the system at the forcing frequency ω

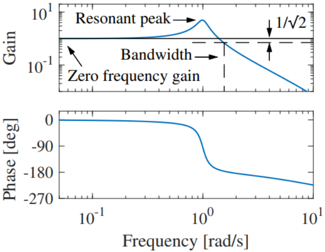
Frequency response - steady-state component

The steady-state solution for a sinusoid $u = \cos \omega t$ is given by

$$y_{\text{ss}}(t) = |G(i\omega)| \cos(\omega t + \angle G(i\omega))$$



(a) Input/output response



(b) Frequency response

Figure 6.11: Steady-state response of an asymptotically stable linear system to a sinusoid. (a) A sinusoidal input of magnitude A_u (dashed) gives a sinusoidal output of magnitude A_y (solid), delayed by ΔT seconds. (b) Frequency response, showing gain and phase. The gain is given by the ratio of the output amplitude to the input amplitude, $M = A_y/A_u$. The phase lag is given by $\theta = -2\pi\Delta T/T$; it is negative for the case shown because the output lags the input.

Some terminology

- **Zero frequency gain**: The gain of a system at ω, corresponds to the ratio between a constant input and the steady output

$$M_0 = G(0) = -CA^{-1}B + D.$$

  - The zero frequency gain is well defined only if A is invertible.
  - Zero frequency gain is a relevant quantity only for **stable systems**.
  - In EE, the zero frequency gain is often called the **DC gain**.

- The **bandwidth** ω_b of a system is the frequency range over which the gain has decreased by no more than a factor $1/\sqrt{2}$ from its reference value (either *zero-frequency gain* or *high-frequency gain*)

- **Resonant peak** M_r, the largest value of the frequency response

- **Peak frequency** ω_mr, the frequency where the maximum occurs
  - The frequency of the sinusoidal input that produces the largest possible output and the gain at the frequency.

Outline

Transfer functions

Transfer functions: transmission of exponential signals e^{st} with $s = \sigma + i\omega$

$$e^{st} = e^{\sigma t} e^{i\omega t} = e^{\sigma t}(\cos \omega t + i \sin \omega t)$$

where $\sigma \leq 0$: decay rate.

- Find the transfer function for the state-space system

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du. \tag{3}$$

- The output $y(t)$ of system (3) to the input e^{st} is

$$y(t) = \underbrace{Ce^{At}\left(x(0) - (sI - A)^{-1}B\right)}_{\text{transient}} + \underbrace{\left(C(sI - A)^{-1}B + D\right)e^{st}}_{\text{steady-state}}$$

- The **transfer function** from u to y of the system (3) is the coefficient of the term e^{st}, i,e,,

$$G(s) = C(sI - A)^{-1}B + D.$$

Example: calculating transfer function

Example

Consider an LTI system

$$\dot{x}_1 = -a_1 x_1 - a_2 x_2 + u \qquad y = x_2$$
$$\dot{x}_2 = x_1$$

- The system matrices are

$$A = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}, D = 0.$$

- Compute its transfer function

$$G(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s + a_1 & a_2 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{s^2 + a_1 s + a_2} \begin{bmatrix} s & -a_2 \\ 1 & s + a_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \frac{1}{s^2 + a_1 s + a_2}.$$

Example: computing steady-state responses

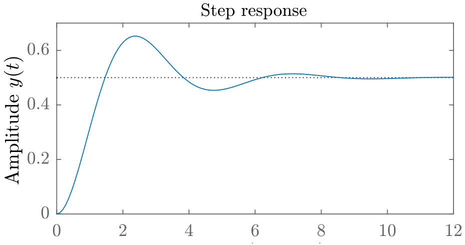
Example

- Suppose $a_1 = 1, a_2 = 2$. Its transfer function is

$$G(s) = \frac{1}{s^2 + s + 2}.$$

- The steady-state response to a step input $u(t) = 1$ is e^{st} with $s = 0$, i.e.

$$y_{\mathrm{ss}} = G(0)u = \frac{1}{2}.$$



Example: computing steady-state responses

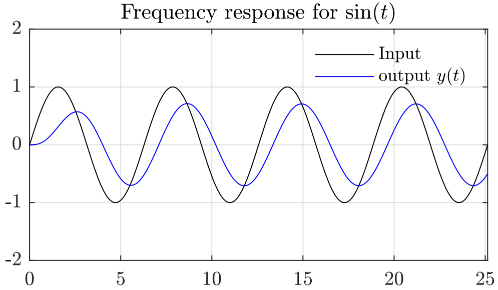
Example

- The steady-state response to a sin input $u(t) = \sin \omega t$ is

$$y = M \sin(\omega t + \theta), \quad \text{where} M = |G(i\omega)|, \theta = \arg(G(i\omega))$$

- **Case 1**: $u(t) = \sin t \ \rightarrow \ y(t)$?

$$G(i\omega) = \frac{1}{(i\omega)^2 + i\omega + 2}, \qquad M = |G(i)| = \frac{1}{\sqrt{2}}, \quad \theta = -45^\circ$$



Example: computing steady-state responses

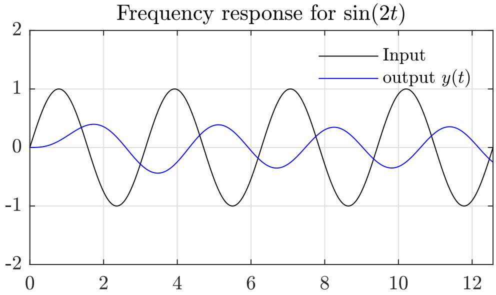
Example

- The steady-state response to a sin input $u(t) = \sin \omega t$ is

$$y = M \sin(\omega t + \phi), \quad \text{where} M = |G(i\omega)|, \theta = \arg(G(i\omega))$$

- **Case 2**: $u(t) = \sin 2t \;\rightarrow\; y(t)$?

$$G(i2) = \frac{1}{-2 + i2}, \qquad M = |G(i2)| = \frac{1}{2\sqrt{2}}, \quad \theta = -135^\circ$$

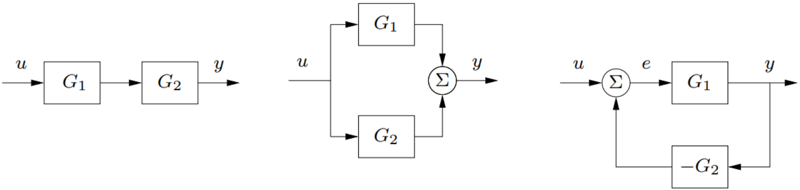


Outline

Block diagrams

The combination of block diagrams and transfer functions is a powerful way to represent control systems.

- Input-output relationship can be derived by **algebraic manipulations** of the transfer functions.



(a) $G_{yu} = G_2 G_1$ (b) $G_{yu} = G_1 + G_2$ (c) $G_{yu} = \dfrac{G_1}{1 + G_1 G_2}$

Figure: Interconnections of linear systems. **Series** (a), **parallel** (b), and **feedback** (c) connections are shown.

Feedback connection

- It is easy to see the relationship

$$y = G_1 e, \qquad e = u - G_2 y$$

- Elimination of e gives

$$\begin{aligned} y = G_1(u - G_2 y) \quad &\Rightarrow \quad (1 + G_1 G_2) y = G_1 u \\ &\Rightarrow \quad y = \frac{G_1}{1 + G_1 G_2} u \end{aligned}$$

- The transfer function of the feedback connection is thus

$$G = \frac{G_1}{1 + G_1 G_2}.$$

These three basic interconnections can be used as the basis for computing transfer functions for more complicated systems.

Outline

Summary

- Transient response and steady-state response

$$y(t) = \underbrace{Ce^{At}\left(x(0) - (sI - A)^{-1}B\right)}_{\text{transient}} + \underbrace{\left(C(sI - A)^{-1}B + D\right)e^{st}}_{\text{steady-state}}$$

- Transfer function

$$G(s) = C(sI - A)^{-1}B + D.$$

  - Express the steady-state solution of a stable linear system forced by a sinusoidal input

$$u(t) = \sin(\omega t) \to y_{\text{ss}} = |G(i\omega)| \sin(\omega t + \angle G(i\omega))$$

- **Frequency domain modeling**: Modeling a system through its response to sinusoidal and exponential signals.
  - We represent the dynamics of the system in terms of the generalized frequency s rather than the time domain variable t.
  - The **transfer function** provides a complete representation of a linear system in the frequency domain.