ECE 171A: Linear Control System Theory
Lecture 13: Transfer functions (II)

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May 01, 2024

Reading materials: Ch 9.2, 9.3, 9.4

Announcements

- **Midterm I**: Many of you did very well (send us an email or come to office hours if you would like to chat)
  - Maximum: 101.5
  - Median: 86
  - Mean: 80
- Anonymous survey on Midterm 1 feedback; Please spend 2 minutes filling it out by Thursday night.

  `https://forms.gle/U838wkGwqCmvCMaj7`
- **Office hours**
  - Ideally, I would like most of you, if not all, to go to the office hours together even if you don't have questions. You can even help us answer questions from others. It is important to have a supportive community for this class!
- **Piazza**: Check it regularly, and feel free to ask questions (Lectures, textbook, HW etc.)

Outline

Frequency-domain modeling

Transfer functions for linear ODEs

Block Diagrams and Transfer Functions

Summary

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Convolution equation and Transfer functions

Consider a state-space system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$.

- The solution to the state-space system (1) is given by

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t). \tag{2}$$

Transfer functions:

- We apply the convolution equation to $u = e^{st}$

$$y(t) = \underbrace{Ce^{At}\left(x(0) - (sI - A)^{-1}B\right)}_{\text{transient}} + \underbrace{\left(C(sI - A)^{-1}B + D\right)e^{st}}_{\text{steady-state}}$$

- The transfer function for the state-space system (1) is

$$G(s) = C(sI - A)^{-1}B + D$$

Transfer functions - overview

- **Transfer functions** — A compact description of the input/output relation for a linear time-invariant (LTI) system.
- Combining transfer functions with **block diagrams** gives a powerful algebraic method to analyze linear systems with many blocks.

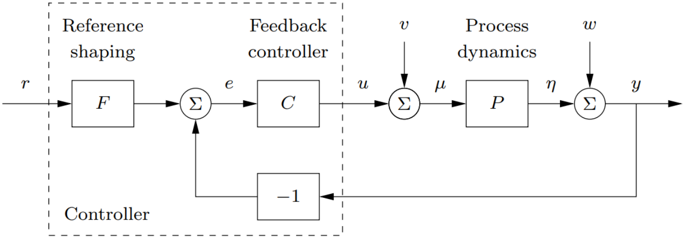

Figure: A block diagram for a feedback control system

- The transfer function allows new interpretations of system dynamics.
- **Many graphical tools**, such as the Bode plot (a powerful graphical representation of the transfer function that was introduced by Bode.)

Response to periodic inputs

The basic idea of the transfer function comes from looking at the frequency response of a system.

$$G(s) = C(sI - A)^{-1}B + D \quad \leftarrow \quad \textbf{Transfer function}$$

- Suppose that we have an input signal that is periodic. We can then decompose it

$$u(t) = \sum_{k=0}^{\infty} \left(a_k \sin(k\omega_\mathrm{f} t) + b_k \cos(k\omega_f t)\right)$$

- The output will be sine and cosine waves, with possibly **shifted magnitude and phase**, which can be determined by

$$G(i\omega) = C(i\omega - A)^{-1}B + D,$$

  where $\omega = k\omega_\mathrm{f}, k = 1, \ldots, \infty$.

- Thanks to linearity (**superposition**), the final steady-state response will be a sum of these signals.

The exponential input e^{st}

The transfer function generalizes this notion to allow a broader class of input signals besides periodic ones.

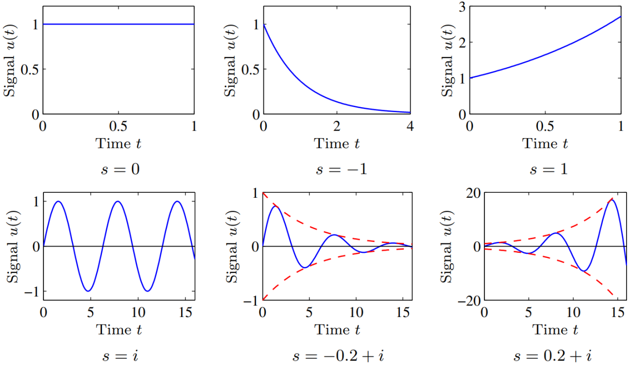

Figure: Examples of exponential signals. The top row: exponential signals with a real exponent, and the bottom row: those with complex exponents.

- The transfer function can also be introduced as the ratio of the *Laplace transforms* of the output and the input.

Frequency-domain modeling

Frequency domain modeling: Modeling a system through its response to sinusoidal and exponential signals.

- We represent the dynamics of the system in terms of the generalized frequency s rather than the time domain variable t.
- The transfer function provides a complete representation of a linear system in the frequency domain.

Some benefits of transfer functions:

- Provide a particularly **convenient representation** in manipulating and analyzing complex linear feedback systems.
- **Graphical representations** (Bode and Nyquist plots) that capture interesting properties of the underlying dynamics — Weeks 5/6
- We can introduce concepts that express the degree of stability of a system – **stability margins**, Week 6
- Express the changes/uncertainty in a system because of modeling error, considering sensitivity to process variations – **robustness**, Week 9

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Linear ODEs

Consider a linear system described by the controlled differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \ldots + b_0 u, \quad (3)$$

where u is the input, and y is the output.

- We aim to determine the transfer function of (3) (i.e., input/output relationship in frequency domain);
- Let the input $u(t) = e^{st}$, and since the system is linear, the output is $y(t) = y_0 e^{st}$.
- Plug $u(t) = e^{st}$ and $y(t) = y_0 e^{st}$ into (3),

$$(s^n + a_{n-1}s^{n-1} + \ldots + a_0)y_0 e^{st} = (b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0)e^{st}$$

- We now have

$$y(t) = y_0 e^{st} = \underbrace{\frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}}_{G(s)} e^{st}.$$

Example: Cruise control

Example

The system dynamics are given by

$$\dot{p} = v(t), \qquad \dot{v}(t) = \frac{1}{m}u(t).$$

where p denotes the position, v denotes the velocity of the vehicle.

- It is the same as
$$\ddot{p} = \frac{1}{m}u(t).$$
- Applying an exponential input $u = e^{st}$ leads to
$$s^2 p_0 e^{st} = \frac{1}{m}e^{st} \qquad \Rightarrow \qquad s^2 y(t) = \frac{1}{m}u(t).$$
- The input/output relationship between $p(t)$ and $u(t)$ (i.e., transfer function) in the frequency domain is
$$G(s) = \frac{1}{ms^2}.$$

Example: spring-mass system

Example

The system dynamics are given by

$$\ddot{x}(t) + 2\zeta\omega_0\dot{x}(t) + \omega_0^2 x(t) = u(t),$$

where $x(t)$ denotes the position of the mass, ζ is the damping coefficient, and ω_0 denotes the natural frequency.

- Applying an exponential input $u = e^{st}$ leads to

$$\begin{aligned} & s^2 x_0 e^{st} + 2\zeta\omega_0 s x_0 e^{st} + \omega_0^2 x_0 e^{st} = e^{st} \\ \Rightarrow \quad & (s^2 + 2\zeta\omega_0 s + \omega_0^2) x(t) = u(t). \end{aligned}$$

- The input/output relationship between $x(t)$ and $u(t)$ (i.e., transfer function) in the frequency domain is

$$G(s) = \frac{1}{s^2 + 2\zeta\omega_0 s + \omega_0^2}.$$

Example: Vibration damper

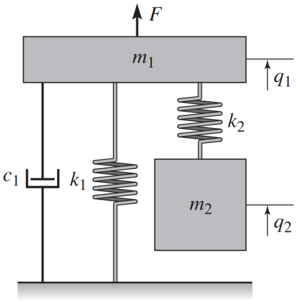


Figure: A vibration damper. Vibrations of the mass m_1 can be damped by providing it with an auxiliary mass m_2, attached to m_1 by a spring with stiffness k_2. The parameters m_2 and k_2 are chosen so that the frequency $\sqrt{k_2/m_2}$ matches the frequency of the vibration.

Example: Vibration damper

Example

The system dynamics are given by

$$\begin{aligned} m_1\ddot{q}_1 + c_1\dot{q}_1 + k_1 q_1 + k_2(q_1 - q_2) &= F, \\ m_2\ddot{q}_2 + k_2(q_2 - q_1) &= 0. \end{aligned}$$

- **Objective**: determine the transfer function from the force F to the position q_1.
- We first find particular exponential solutions

$$\begin{aligned} (m_1 s^2 + c_1 s + k_1)q_1 + k_2(q_1 - q_2) &= F \\ m_2 s^2 q_2 + k_2(q_2 - q_1) &= 0. \end{aligned}$$

- Eliminate q_2 and we have the transfer function

$$G_{q_1 F}(s) = \frac{m_2 s^2 + k_2}{m_1 m_2 s^4 + m_2 c_1 s^3 + (m_1 k_2 + m_2(k_1 + k_2))s^2 + k_2 c_1 s + k_1 k_2}$$

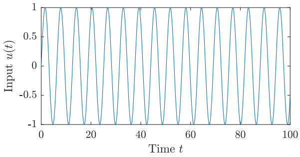
- The transfer function has a **zero** at $s = \pm i\sqrt{k_2/m_2}$ **— Blocking property**

Blocking property

Parameters $m_1 = 1, c_1 = 1, k_1 = 1, m_2 = 1, k_2 = 1.$

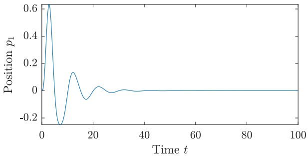
- **Case 1: external input**

$$u = \sin(\omega t), \qquad \text{with } \omega = 1.$$

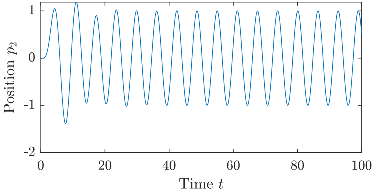


(a) Input $u = \sin(t)$

$\Rightarrow$



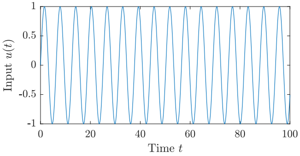
(b) Position of mass 1



(c) Postion of mass 2

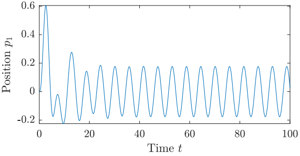
Other frequencey responses

- **Case 2: external input** $u = \sin(\omega t)$, with $\omega = 1.1$.



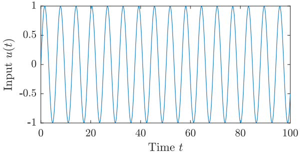
(a) Input $u = \sin(1.1t)$

$\Longrightarrow$



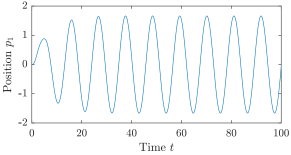
(b) Position of mass 1

- **Case 3: external input** $u = \sin(\omega t)$, with $\omega = 0.578$.



(a) Input $u = \sin(1.1t)$

$\Longrightarrow$



(b) Position of mass 1

Common transfer functions

Type	System	Transfer function
Integrator	$\dot{y} = u$	$\frac{1}{s}$
Differentiator	$y = \dot{u}$	s
First-order system	$\dot{y} + ay = u$	$\frac{1}{s+a}$
Double integrator	$\ddot{y} = u$	$\frac{1}{s^2}$
Damped oscillator	$\ddot{y} + 2\zeta\omega_0\dot{y} + \omega_0^2 y = u$	$\frac{1}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$
State-space system	$\dot{x} = Ax + Bu$ $y = Cx + Du$	$C(sI - A)^{-1}B + D$
PID controller	$y = k_\mathrm{p} u + k_\mathrm{d}\dot{u} + k_\mathrm{i}\int u$	$k_\mathrm{p} + k_\mathrm{d} s + \frac{k_\mathrm{i}}{s}$
Time delay	$y(t) = u(t-\tau)$	$e^{-\tau s}$

Table: Transfer functions for some common linear time-invariant systems.

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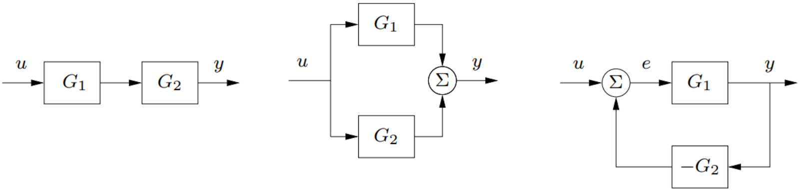
Block diagrams

The combination of block diagrams and transfer functions is a powerful way to represent control systems.

- Input-output relationship can be derived by **algebraic manipulations** of the transfer functions.



(a) $G_{yu} = G_2 G_1$ (b) $G_{yu} = G_1 + G_2$ (c) $G_{yu} = \dfrac{G_1}{1 + G_1 G_2}$

Figure: Interconnections of linear systems. **Series** (a), **parallel** (b), and **feedback** (c) connections are shown.

Feedback connection

- It is easy to see the relationship

$$y = G_1 e, \qquad e = u - G_2 y$$

- Elimination of e gives

$$\begin{aligned} y = G_1(u - G_2 y) \quad &\Rightarrow \quad (1 + G_1 G_2)y = G_1 u \\ &\Rightarrow \quad y = \frac{G_1}{1 + G_1 G_2} u \end{aligned}$$

- The transfer function of the feedback connection is thus

$$G = \frac{G_1}{1 + G_1 G_2}.$$

These three basic interconnections can be used as the basis for computing transfer functions for more complicated systems.

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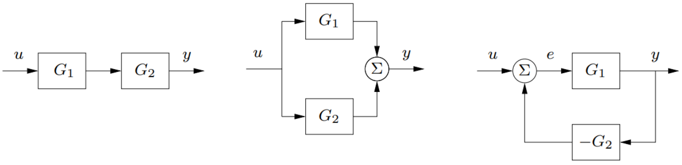
- **Frequency domain modeling**: Modeling a system through its response to sinusoidal and exponential signals.
  - We represent the dynamics of the system in terms of the generalized frequency s rather than the time domain variable t.
  - The **transfer function** provides a complete representation of a linear system in the frequency domain.
- **Transfer function for linear ODEs**

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \ldots + b_0 u,$$

$$G(s) = \frac{b_m s^m + b_{m-1}s^{n-1} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}.$$

- **Block diagram with transfer functions**



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