ECE 171A: Linear Control System Theory
Lecture 14: Zeros, Poles and Bode plot

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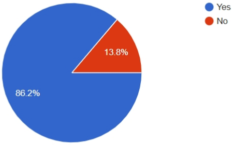
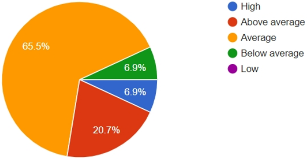
Reading materials: Ch 9.5, 9.6

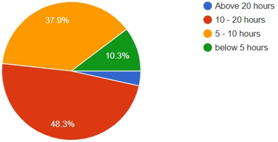
Survey Feedback



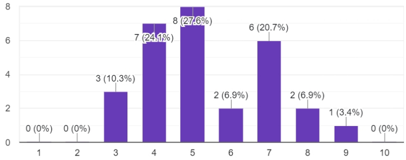
(a) First Control course



(b) Work load



(c) hours each week



(d) Difficulty - Midterm 1

Survey Feedback

Q: Which aspect(s) of this course have you particularly enjoyed or valued so far? Any other comments on the course [1]

- "Representative hw and exams"
- "The structure of lectures are very organized and clear."
- "The professor is a very nice person (creating a welcoming environment both in class and in office hours). The course settings, lecture slides, and the homework description is all very clear. The professor answers questions on piazza on time."
- "HW and office hours have been engaging and fun to work on. Practice exam and lecture 9 materials were super helpful to get a grasp of what we had."
- "The professor is cool and I like his lecture structures and his teaching style"
- "I enjoy learning about how to determine the stability of a system using mathematic methods."
- ......

[1] These are copied from the answers. If you do not want them to be here, I'll remove them.

Survey Feedback

Q: Which aspect(s) of this course do you think could be improved or changed for the rest of this quarter?[2]

- Nothing. Best course I've ever took.
- I think so far everything looks good
- More homework instructions
- I feel lectures do not include a lot of examples. Or they go through them too fast, but I understand the time constraint
- More questions/examples during class more similar to the questions we see in the homework.
- I hope attendance can be bonus credits
- I am interested in learning about more system analysis techniques, and not about how specific systems behave (e.g. predator-prey, inverted pendulum).
- ...

[2] These are adapted from the answers. If you do not want them to be here, I'll remove them.

Outline

Outline

Transfer functions

Type	System	Transfer function
Integrator	$\dot{y} = u$	$\frac{1}{s}$
Differentiator	$y = \dot{u}$	s
First-order system	$\dot{y} + ay = u$	$\frac{1}{s+a}$
Double integrator	$\ddot{y} = u$	$\frac{1}{s^2}$
Damped oscillator	$\ddot{y} + 2\zeta\omega_0\dot{y} + \omega_0^2 y = u$	$\frac{1}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$
State-space system	$\dot{x} = Ax + Bu$ $y = Cx + Du$	$C(sI - A)^{-1}B + D$

The **features** of a transfer function are often associated with **important system properties**.

- zero frequency gain
- the locations of the poles and zeros.

Zero frequency gain

Zero frequency gain: the magnitude of the transfer function at $s = 0$.

- **Interpretation**: The steady-state value of the output with respect to a unit step input (which can be represented as $u = e^{st}$ with $s = 0$).

Examples:

- State-space model (steady state of a step response on Page 6 of L12):

$$G(s) = C(sI - A)^{-1}B + D \qquad \Rightarrow \qquad G(0) = D - CA^{-1}B$$

- Linear differential equation:

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0 y = b_1\frac{du}{dt} + b_0 u \qquad \Rightarrow \qquad G(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}$$

$$\Rightarrow \qquad G(0) = \frac{b_0}{a_0}.$$

- Integrator $\dot{y} = u$, $G(s) = \dfrac{1}{s}$: we have $G(0) = \infty$.
- Differentiator $y = \dot{u}$, $G(s) = s$: we have $G(0) = 0$.

Example: computing steady-state responses

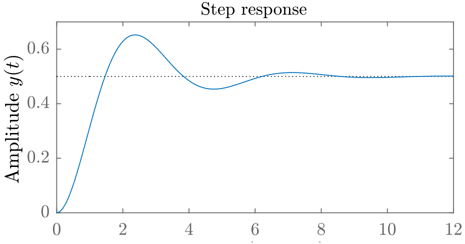
Example

- Consider a transfer function (which has **stable** poles)

$$G(s) = \frac{1}{s^2 + s + 2}.$$

- The steady-state response to a step input $u(t) = 1$ is e^{st} with $s = 0$, i.e.

$$y_{\text{ss}} = G(0)u = \frac{1}{2}.$$



Poles and zeros

Consider a linear system with the rational transfer function

$$G(s) = \frac{b(s)}{a(s)}.$$

- **Poles**: The roots of the polynomial $a(s)$.
- **Zeros**: The roots of the polynomial $b(s)$.

Interpretation of poles: Stability of the system

- **Unstable pole** if $\mathrm{Re}(p) > 0$; **Stable pole** if $\mathrm{Re}(p) < 0$.
- Consider a linear differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}y}{dt^{m-1}} + \ldots + b_0 u. \quad (1)$$

- Let $u = 0$ (no external force; homogeneous ODE). If p is a pole, i.e., p is a solution to

$$s^n + a_{n-1}s^{n-1} + \ldots + a_0 = 0$$

- then, $y(t) = y_0 e^{pt}$ is a particular solution to (1) for initial response.

Poles and zeros

Interpretation of zeros:

- Consider an exponential input e^{st}
- The exponential output is $y(t) = G(s)e^{st}$.
- If $G(s) = 0$, then the (steady-state) output is zero.

> Zeros of a **stable** transfer function thus **block** transmission of the corresponding exponential signals.

Example (Vibration dampers)

$$G_{q_1 F}(s) = \frac{m_2 s^2 + k_2}{m_1 m_2 s^4 + m_2 c_1 s^3 + (m_1 k_2 + m_2(k_1 + k_2))s^2 + k_2 c_1 s + k_1 k_2}$$

- The transfer function has a **zero** at $s = \pm i\sqrt{k_2/m_2}$ — **Blocking property**

Example: Vibration damper

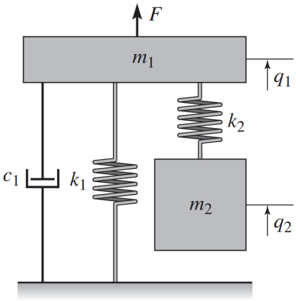


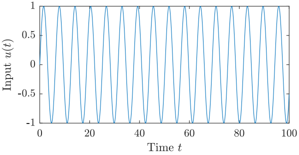
Figure: A vibration damper. Vibrations of the mass m_1 can be damped by providing it with an auxiliary mass m_2, attached to m_1 by a spring with stiffness k_2. The parameters m_2 and k_2 are chosen so that the frequency $\sqrt{k_2/m_2}$ matches the frequency of the vibration.

Blocking property

Parameters $m_1 = 1, c_1 = 1, k_1 = 1, m_2 = 1, k_2 = 1$.

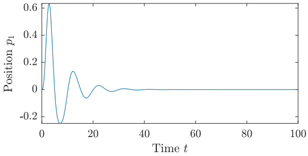
- **The following external input is blocked**; the output of mass 1 becomes zero after some transient

$$u = \sin(\omega t), \qquad \text{with} \;\; \omega = 1.$$

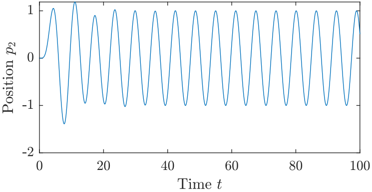


(a) Input $u = \sin(t)$

$\Rightarrow$



(b) Position of mass 1



(c) Postion of mass 2

Pole zero diagram

Pole-zero diagram: A convenient way to view the poles and zeros of a transfer function.

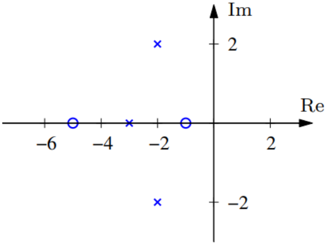


Figure: A pole zero diagram for a transfer function with zeros at -5 and -1 and poles at -3 and $-2 \pm 2j$. The circles represent the locations of the zeros, and the crosses the locations of the poles.

- **Stable poles**: Poles in the left half-plane
- **Unstable poles**: Poles in the right half-plane

Some connections

State-space models vs. **transfer function representations**
(assuming SISO system)

	State-space model	Transfer function
Model	$\dot{x} = Ax + Bu$ $y = Cx + Du$	$G(s) = C(sI - A)^{-1}B + D = \frac{b(s)}{a(s)}$
Variables	input $u(t) \in \mathbb{R}$, output $y(t) \in \mathbb{R}$, state $x(t) \in \mathbb{R}^n$	input $u(t) \in \mathbb{R}$, output $y(t) \in \mathbb{R}$,
Stability	Poles (eigenvalues) of A	Poles of $G(s)$

Poles (eigenvalues) of the matrix A = Poles of the transfer function $G(s)$

- The inverse of $(sI - A)$ can be computed below $\Rightarrow a(s) = \det(sI - A)$.

$$(sI - A)^{-1} = \frac{1}{\det(sI - A)}\text{adj}(sI - A).$$

Outline

Bode plot

The **frequency response** of a linear system can be computed from its transfer function by setting $s = i\omega$, i.e.,

$$u(t) = e^{i\omega t} = \cos(\omega t) + i\sin(\omega t).$$

- The resulting output is

$$y(t) = G(i\omega)e^{i\omega t} = Me^{i(\omega t+\theta)} = M\cos(\omega t + \theta) + iM\sin(\omega t + \theta)$$

- Thus, we have $\cos(\omega t) \to M\cos(\omega t + \theta)$ and $\sin(\omega t) \to M\sin(\omega t + \theta)$

The frequency response $G(i\omega)$ can be represented by two curves — **Bode plot**

- **Gain curve**: gives $|G(i\omega)|$ as a function of frequency ω — log/log scale (traditionally often in dB — $20\log|G(i\omega)|$; but we use $\log|G(i\omega)|$)
- **Phase curve**: gives $\angle G(i\omega)$ as a function of frequency ω — log/linear scale in degrees

Sketching Bode plots

- Part of the popularity of Bode plots is that they are easy to sketch and interpret.
- Since the frequency scale is logarithmic, they cover the behavior of a linear system over a wide frequency range.

Consider a transfer function

$$G(s) = \frac{b_1(s)b_2(s)}{a_1(s)a_2(s)}$$

- **Gain curve**: simply adding and subtracting gains corresponding to terms in the numerator and denominator

$$\log|G(s)| = \log|b_1(s)| + \log|b_2(s)| - \log|a_1(s)| - \log|a_2(s)|.$$

- **Phase curve**: similarly we have

$$\angle G(s) = \angle b_1(s) + \angle b_2(s) - \angle a_1(s) - \angle a_2(s).$$

Bode plot - Blocks

A polynomial can be written as a product of terms of the type

$$k, \qquad s, \qquad s+a, \qquad s^2 + 2\zeta\omega_0 s + \omega_0^2$$

- Sketch Bode diagrams for these terms;
- Complex systems: add the gains and phases of the individual terms

Case 1: $G(s) = s^k$ — Two special cases: $k = 1$, a differentiator; $k = -1$, an integrator

$$\log |G(s)| = k \times \log \omega, \qquad \angle G(i\omega) = k \times 90^\circ$$

- The gain curve is a straight line with slope k, and the phase curve is a constant at $k \times 90^\circ$
- The case when $k = 1$ corresponds to a differentiator and has slope 1 with phase 90°
- The case when $k = -1$ corresponds to an integrator and has slope -1 with phase -90°

Case 1: $G(s) = s^k$

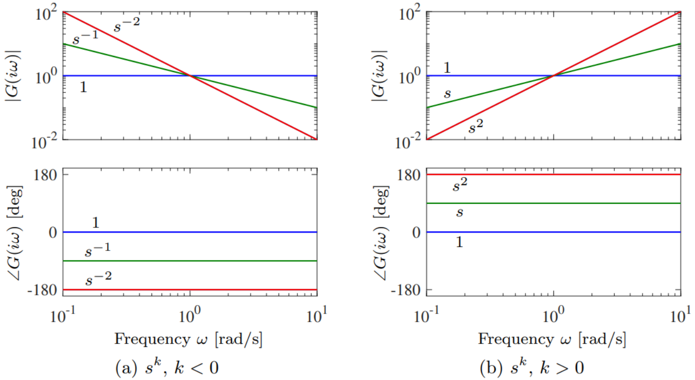
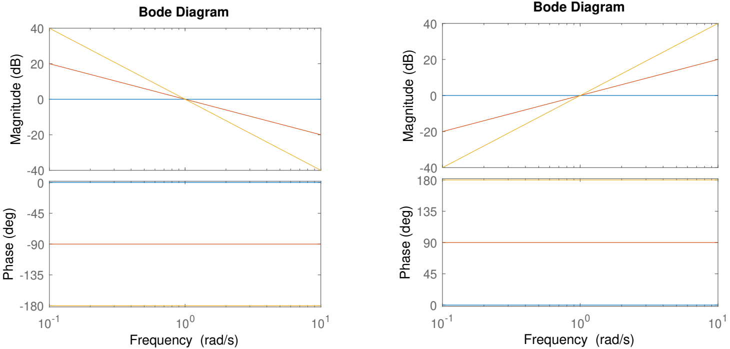


Figure: Bode plots of the transfer functions $G(s) = s^k$ for $k = -2, -1, 0, 1, 2$. On a log-log scale, the gain curve is a straight line with slope k. The phase curves for the transfer functions are constants, with phase equal to $k \times 90°$.

Case 1: $G(s) = s^k$



(a) $s^k, k < 0$ (b) $s^k, k > 0$

Figure: Bode plots of the transfer functions $G(s) = s^k$ for $k = -2, -1, 0, 1, 2$ — from Matlab

```
G0 = tf([1],[1]); % create a transfer function
G1 = tf([1 0],[1]); % create a transfer function
W = {0.1,10}; bode(G0,G1,W); % Bode plot
```

Outline

Summary

- The **features** of a transfer function are often associated with **important system properties**.
  - zero frequency gain
  - the locations of the poles and zeros: Poles — stability of a system; Zeros – Block transmission of certain signals

Poles (eigenvalues) of the matrix A = Poles of the transfer function $G(s)$

- The frequency response $G(i\omega)$ can be represented by two curves — **Bode plot**
  - **Gain curve**: gives $|G(i\omega)|$ as a function of frequency ω — log/log scale (often in dB — $20 \log |G(i\omega)|$)
  - **Phase curve**: gives $\angle G(i\omega)$ as a function of frequency ω — log/linear scale in degrees