ECE 171A: Linear Control System Theory
Lecture 15: Bode plot

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Reading materials: Ch 9.6, Ch 2.2

Outline

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Bode plot

The **frequency response** of a **stable** linear system can be computed from its transfer function by setting $s = i\omega$, i.e.,

$$u(t) = e^{i\omega t} = \cos(\omega t) + i\sin(\omega t).$$

- The resulting steady-state output is

$$y(t) = G(i\omega)e^{i\omega t} = Me^{i(\omega t+\theta)} = M\cos(\omega t+\theta) + iM\sin(\omega t+\theta)$$

- Thus, we have $\cos(\omega t) \to M\cos(\omega t+\theta)$ and $\sin(\omega t) \to M\sin(\omega t+\theta)$

The frequency response $G(i\omega)$ can be represented by two curves — **Bode plot**

- **Gain curve**: gives $|G(i\omega)|$ as a function of frequency ω — log/log scale (traditionally often in dB — $20\log|G(i\omega)|$; but we mainly use $\log|G(i\omega)|$)
- **Phase curve**: gives $\angle G(i\omega)$ as a function of frequency ω — log/linear scale in degrees

Sketching Bode[1] plots

Hendrik Wade Bode (1905 - 1982): a pioneer of modern control theory and electronic telecommunications.

- Part of the popularity of Bode plots is that they are easy to sketch and interpret.
- Since the frequency scale is logarithmic, they cover the system behavior over a wide frequency range.

Consider a transfer function

$$G(s) = \frac{b_1(s)b_2(s)}{a_1(s)a_2(s)}$$

- **Gain curve**: simply adding and subtracting gains corresponding to terms in the numerator and denominator

$$\log|G(s)| = \log|b_1(s)| + \log|b_2(s)| - \log|a_1(s)| - \log|a_2(s)|.$$

- **Phase curve**: similarly we have

$$\angle G(s) = \angle b_1(s) + \angle b_2(s) - \angle a_1(s) - \angle a_2(s).$$

[1] https://en.wikipedia.org/wiki/Hendrik_Wade_Bode

Bode plot — Blocks

A polynomial can be written as a product of terms of the type

$$k, \qquad s, \qquad s+a, \qquad s^2 + 2\zeta\omega_0 s + \omega_0^2$$

- Sketch Bode diagrams for these terms;
- Complex systems: add the gains and phases of the individual terms

Case 1: $G(s) = s^k$ — Two special cases: $k = 1$, a differentiator; $k = -1$, an integrator

$$\log|G(s)| = k \times \log\omega, \qquad \angle G(i\omega) = k \times 90^\circ$$

- The gain curve is a straight line with slope k, and the phase curve is a constant at $k \times 90^\circ$
- The case when $k = 1$ corresponds to a differentiator and has slope 1 with phase 90°
- The case when $k = -1$ corresponds to an integrator and has slope -1 with phase -90°

Case 1: $G(s) = s^k$

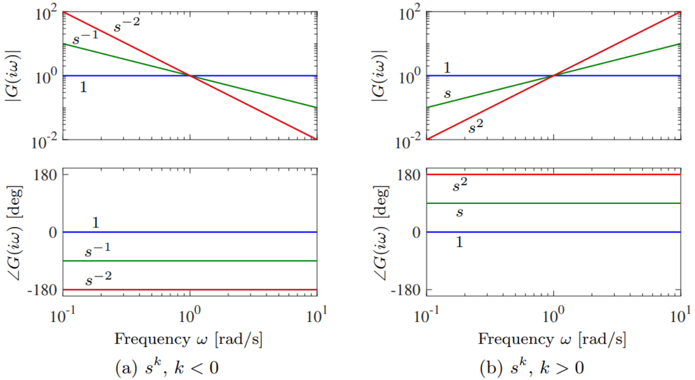
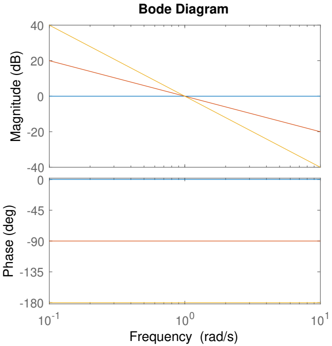


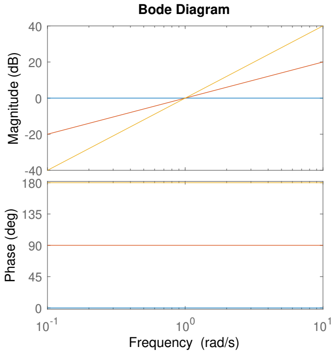
Figure: Bode plots of the transfer functions $G(s) = s^k$ for $k = -2, -1, 0, 1, 2$. On a log-log scale, the gain curve is a straight line with slope k. The phase curves for the transfer functions are constants, with phase equal to $k \times 90°$.

Case 1: $G(s) = s^k$

```
G0 = tf([1],[1]); % create a transfer function
G1 = tf([1 0],[1]); % create a transfer function
W = {0.1,10}; bode(G0,G1,W); % Bode plot
```



(a) $s^k, k < 0$



(b) $s^k, k > 0$

Figure: Bode plots of the transfer functions $G(s) = s^k$ for $k = -2, -1, 0, 1, 2$ — from Matlab

Case 2: first-order system

Consider the transfer function of a first-order system

$$G(s) = \frac{a}{s+a}, \qquad a > 0.$$

- We have

$$|G(s)| = \frac{|a|}{|s+a|}, \qquad \angle G(s) = \angle a - \angle(s+a).$$

- The gain curve is

$$|G(i\omega)| = \log a - \frac{1}{2}\log(\omega^2 + a^2) \approx \begin{cases} 0, & \text{if } \omega < a \\ \log a - \log \omega, & \text{if } \omega > a \end{cases}$$

- The phase curve is

$$\angle G(i\omega) = -\frac{180}{\pi}\arctan\frac{\omega}{a} \approx \begin{cases} 0, & \text{if } \omega < \frac{a}{10} \\ -45 - 45(\log\omega - \log a), & \text{if } a/10 < \omega < 10a \\ -90, & \text{if } \omega > 10a \end{cases}$$

Case 2: first-order system

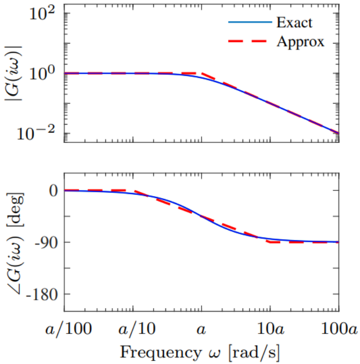


Figure: Bode plot of the first-order system $G(s) = a/(s+a)$, which can be approximated by asymptotic curves (dashed) in both the gain and the frequency, with the breakpoint in the gain curve at $\omega = a$ and the phase decreasing by 90° over a factor of 100 in frequency.

A first-order system behaves like a constant for low frequencies and like an integrator for high frequencies.

Case 3: second-order system

Consider the transfer function of a first-order system

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}, \qquad 0 < \zeta < 1.$$

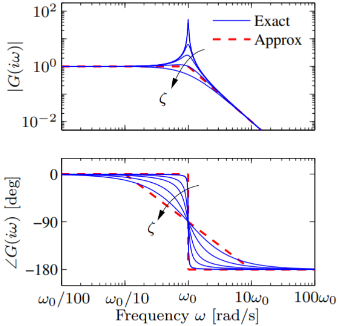
- The gain curve is

$$\begin{aligned} |G(i\omega)| &= 2\log\omega_0 - \frac{1}{2}\log(\omega^4 + 2\omega_0^2\omega^2(2\zeta^2 - 1) + \omega_0^4) \\ &\approx \begin{cases} 0, & \text{if } \omega \ll \omega_0 \\ 2\log\omega_0 - 2\log\omega, & \text{if } \omega \gg \omega_0 \end{cases} \end{aligned}$$

- The largest gain $Q = \max_\omega |G(i\omega)| \approx 1/(2\zeta)$, called the Q-value, is obtained for $\omega \approx \omega_0$ – **Resonant frequency**
- The phase curve is

$$\angle G(i\omega) = -\frac{180}{\pi}\arctan\frac{2\zeta\omega_0\omega}{\omega_0^2 - \omega} \approx \begin{cases} 0, & \text{if } \omega \ll \omega_0 \\ -180, & \text{if } \omega \gg \omega_0 \end{cases}$$

Case 3: Second-order system



(b) Second-order system

Figure: Bode plot of the second-order system $G(s) = \omega_0^2/(s^2 + 2\zeta\omega_0 s + \omega_0^2)$, which has a peak at frequency ω_0 and then a slope of -2 beyond the peak; the phase decreases from 0° to -180°. The height of the peak and the rate of change of phase depend on the damping ratio ζ ($\zeta = 0.02, 0.1, 0.2, 0.5$, and 1.0 shown).

The asymptotic approximation is poor near $\omega = \omega_0$ and the Bode plot depends strongly on ζ near this frequency. — **More examples in Discussion 6**

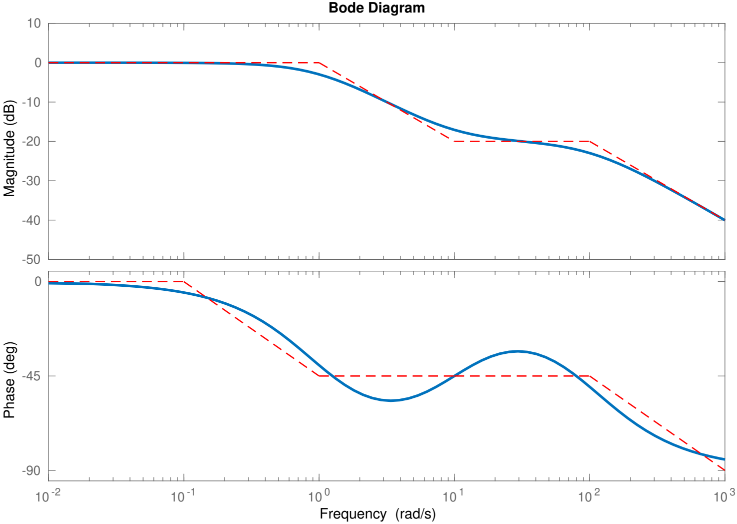
High-order Example

Example

Draw a Bode plot for $G(s) = 10\frac{s+10}{(s+1)(s+100)}$

- Step 1: find break points (related to poles and zeros): 1, 10, 100.
- Step 2: Calculate $|G(i0)|$ and $\angle G(i0)$ to determine the starting points
- Step 3: Sketch the bode plot by the rules
  - **Magnitude increases with a zero**: if the zero is a first-order real zero, the slop is $+1$; if the zero is a second-order zero (or complex zero), the slop is $+2$
  - **Magnitude decreases with a pole**: If there pole is a first-order real pole, the slop is -1; if the pole is a second-order pole (or complex pole), the slop is -2
  - **Phases changes** by $+90$ with a first order real zero; $+180$ with a second order zero (or complex zero). The change starts around $a/10$ and ends around $10a$.
  - **Phases changes** by -90 with a first order real pole; -180 with a second order pole (or complex pole). Similarly, the change starts around $a/10$ and ends around $10a$.

High-order Example



Outline

System insights

The Bode plot gives a quick overview of a **stable** linear system. Since many useful signals can be decomposed into a sum of sinusoids, it is possible to *visualize* the behavior of a system for different frequency ranges.

$$u(t) = \sin(\omega t) \qquad \rightarrow \qquad y_{\text{ss}} = |G(i\omega)| \sin(\omega t + \angle G(i\omega))$$

- The system can be viewed as a **filter**: change the input signals according to frequency range
- **Type 1: Lower-pass filter**, for example

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

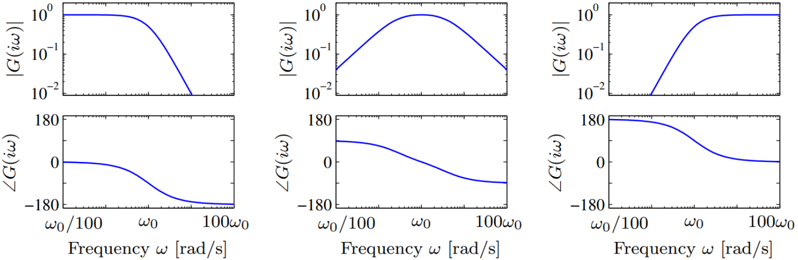
- **Type 2: Band-pass filter**, for example

$$G(s) = \frac{2\zeta\omega_0 s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

- **Type 3: High-pass filter**, for example

$$G(s) = \frac{s^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

Filters



$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(a) Low-pass filter

$$G(s) = \frac{2\zeta\omega_0 s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(b) Band-pass filter

$$G(s) = \frac{s^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(c) High-pass filter

Figure: Bode plots for low-pass, band-pass, and high-pass filters. The upper plots are the gain curves and the lower plots are the phase curves. Each system passes frequencies in a different range and attenuates frequencies outside of that range.

Example: Spring-mass system

Example

Consider a spring-mass with input u (force) and output q (position) as follows

$$m\ddot{q} + c\dot{q} + kq = u \qquad \rightarrow \qquad G(s) = \frac{1}{ms^2 + cs + k}$$

- **Case 1**: When s is small, we have

$$G(s) \approx \frac{1}{k} \qquad \rightarrow \qquad q = \frac{u}{k}$$

  which implies that for low-frequency inputs, the system behaves like a **spring** driven by a force.

- **Case 2**: When s is large, we have

$$G(s) \approx \frac{1}{ms^2} \qquad \rightarrow \qquad \ddot{q} = \frac{u}{m}$$

  which implies that the system behaves like a **mass** driven by a force (double integrator).

Example: Spring-mass system

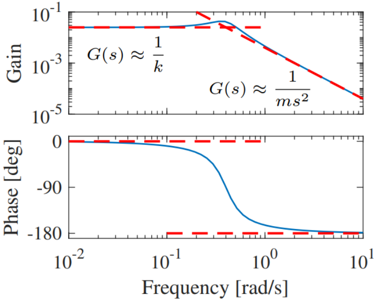
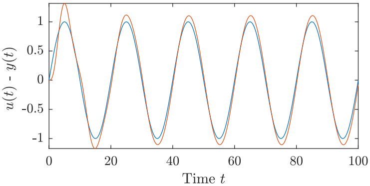


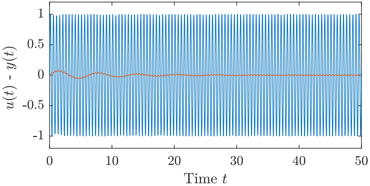
Figure: Bode plot for a spring–mass system. At low frequency the system behaves like a spring with $G(s) \approx 1/k$ and at high frequency the system behaves like a pure mass with $G(s) \approx 1/(ms^2)$

Example: Spring-mass system

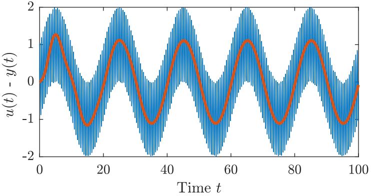
Consider parameters $m = 1; k = 1; c = 0.2;$



(a) Low frequency



(b) High frequency

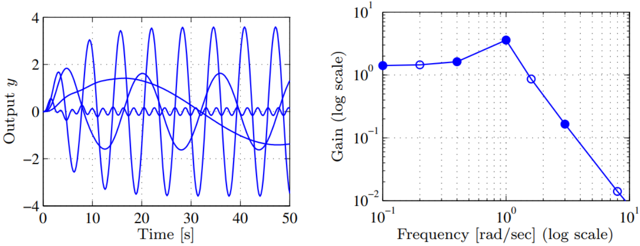


(c) Mixed frequencies

Determine Transfer function experimentally

Model a given application by measuring the frequency response

- Apply a sinusoidal signal at a fixed frequency.
- Measure the amplitude ratio and phase lag when steady state is reached.
- The complete frequency response is obtained by sweeping over a range of frequencies.



(a) Time domain simulations (b) Frequency response

Figure: A frequency response (gain only) computed by measuring the response of individual sinusoids.

Outline

Summary

- The frequency response $G(i\omega)$ can be represented by two curves — **Bode plot**
  - **Gain curve**: gives $|G(i\omega)|$ as a function of frequency ω — log/log scale (traditionally often in dB — $20 \log |G(i\omega)|$; but we mainly use $\log |G(i\omega)|$)
  - **Phase curve**: gives $\angle G(i\omega)$ as a function of frequency ω — log/linear scale in degrees;
- **Empirical rules** of sketching Bode plots
- System insights from Bode plots
  - A system can be viewed as a **filter**: change the input signals according to the frequency range. **Low-pass filter**, **band-pass filter**, **high-pass filter**;
  - Determine a transfer function from experiments