ECE 171A: Linear Control System Theory

Lecture 16: Routh–Hurwitz Criterion and Loop transfer functions

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May 08, 2024

Reading materials: Ch 10.1, 10.2

Outline

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Stability

Theorem (Stability of a linear system (Lyapunov sense))

The system $\dot{x} = Ax$ is

- **asymptotically stable** *if and only if all eigenvalues of A have a strictly negative real part, i.e.,* $\mathrm{Re}(\lambda_i) < 0$
- **unstable** *if any eigenvalues A has a strictly positive real part.*

Consider an LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du \end{aligned} \qquad \Longleftrightarrow \qquad G(s) = C(sI - A)^{-1}B + D$$

Poles (eigenvalues) of the matrix A = Poles of the transfer function $G(s)$

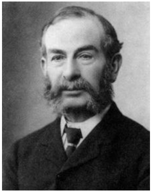
- A system is **bounded-input bounded-output (BIBO)** stable if every bounded input $u(t)$ leads to a bounded output $y(t)$.
- **BIBO stable**: if all poles of $G(s)$ are in the open left half-plane in the s domain (i.e., having negative real parts).

Routh-Hurwitz Criterion

- **Eigenvalues** or **poles**

$$G(s) = \frac{b(s)}{a(s)}, \qquad a(s) = \det(sI - A)$$

- In the 1870s-1890s, **Edward Routh** (English Mathematician, 1831 – 1907) and **Adolf Hurwitz** (German Mathematician, 1859 – 1919) independently
  - developed a method for determining the **locations** in the s plane but **not the actual values** of the roots of a polynomial with constant real coefficients



E. Routh

- Characteristic polynomial:

$$a(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0$$

- **The Routh-Hurwitz method**
  - constructs a table with $n+1$ rows from the coefficients a_i of a polynomial $a(s)$
  - relates **the number of sign changes** in the first column of the table to **the number of roots** in the closed right half-plane

A. Hurwitz

Routh Table

- $a(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0$

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$	a_0
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$	0
s^{n-2}	$b_{n-1} = -\frac{\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}}$	$b_{n-3} = -\frac{\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{a_{n-1}}$	b_{n-5}	$\cdots$	0
s^{n-3}	$c_{n-1} = -\frac{\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}}{b_{n-1}}$	$c_{n-3} = -\frac{\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_{n-1} & b_{n-5} \end{vmatrix}}{b_{n-1}}$	c_{n-5}	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$
s^0	a_0	0	0	$\cdots$	0

- Any row can be multiplied by a positive constant without changing the result

Routh-Hurwitz BIBO Stability Criterion

Theorem

Consider a Routh table from the polynomial $a(s)$ in

$$G(s) = \frac{b(s)}{a(s)}.$$

- *The number of sign changes in the first column of the Routh table is equal to the number of roots of $a(s)$ in the closed right half-plane.*

Corollary (BIBO Stability of LTI Systems)

The system $G(s)$ is **BIBO stable** *if and only if there are no sign changes in the first column of its Routh table.*

There are two special cases related to the Routh table:

1. The first element of a row is 0 but some of the other elements are not
   - **Solution**: replace the 0 with an arbitrary small ϵ
2. All elements of a row are 0 (not required in this course)

Example: Second-order System

Example

Consider the characteristic polynomial of a second-order system:

$$a(s) = as^2 + bs + c$$

- The Routh table is:

s^2	a	c
s^1	b	0
s^0	$-\frac{1}{b}(0 - bc) = c$	0

- A **necessary and sufficient condition** for BIBO stability of a second-order system is that all coefficients of the characteristic polynomial are non-zero and have the same sign.

Example: Third-order System

Example

Consider the characteristic polynomial of a third-order system:

$$a(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

- The Routh table is:

s^3	a_3	a_1
s^2	a_2	a_0
s^1	$-\frac{1}{a_2}(a_3 a_0 - a_1 a_2)$	0
s^0	a_0	0

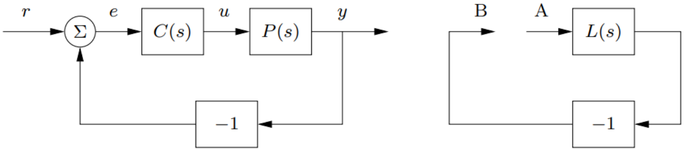
- If $a_3 > 0$, then a **sufficient and necessary condition** for BIBO stability (all eigenvalues have strictly negative real parts) is

$$a_3 > 0, \qquad a_2 > 0, \qquad a_1 a_2 > a_0 a_3, \qquad a_0 > 0$$

- If $a_1 a_2 = a_0 a_3$, one pair of roots lies on the imaginary axis in the s plane and the system is marginally stable. This results in an all zero row in the Routh table.

Outline

Stability of feedback systems



(a) Closed loop system (b) Open loop system

- **Lyapunov stability** — eigenvalue test of the closed-loop matrix; e.g.,

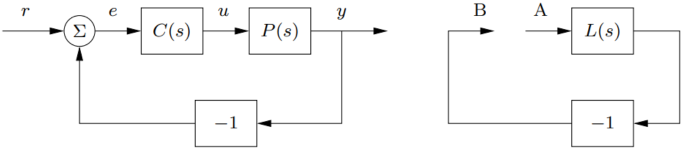
$$\begin{aligned} \text{Dynamics} \rightarrow \quad & \dot{x} = Ax + Bu, \\ \text{Feedback controller} \rightarrow \quad & u = -Kx \end{aligned} \quad \Rightarrow \quad \dot{x} = (A - BK)x.$$

- **Poles or The Routh–Hurwitz Criterion**;

$$\begin{cases} P(s) &= \dfrac{n_{\mathrm{p}}(s)}{d_{\mathrm{p}}(s)} \\ C(s) &= \dfrac{n_{\mathrm{c}}(s)}{d_{\mathrm{c}}(s)} \end{cases} \quad \Rightarrow \quad G_{yr}(s) = \frac{PC}{1+PC} = \frac{n_{\mathrm{p}}(s)n_{\mathrm{c}}(s)}{d_{\mathrm{p}}(s)d_{\mathrm{c}}(s) + n_{\mathrm{p}}(s)n_{\mathrm{c}}(s)}$$

They are **straightforward but give little guidance** for design: it is not easy to tell how the controller should be modified to make an unstable system stable.

Loop analysis



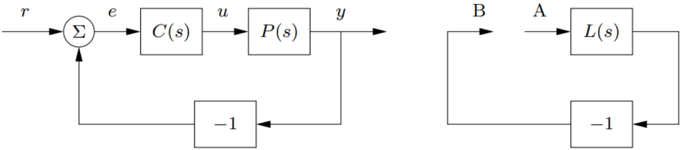
(a) Closed loop system (b) Open loop system

Figure: **The loop transfer function** $L(s) = P(s)C(s)$. The stability of the feedback system (a) can be determined by tracing signals around the loop.

- We break the loop in (b) and ask whether a signal injected at the point A has the same magnitude and phase when it reaches point B.
- Determine **stability and robustness** of closed loop systems by investigating how sinusoidal signals propagate around the feedback loop.
- Reason about the **closed loop behavior** of a system through the frequency domain properties of the **open loop transfer function**.

The second very important graphical tool — the **Nyquist stability theorem**.

Nyquist's idea



(a) Closed loop system (b) Open loop system

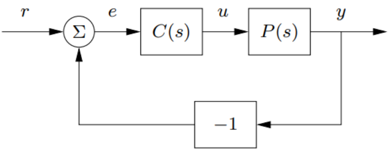
- Nyquist's idea was to first investigate conditions under which oscillations can occur in a feedback loop.
- The **Loop transfer function**:

$$L(s) = P(s)C(s).$$

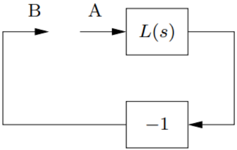
- Assume that a sinusoid of frequency ω_0 is injected at point A. In steady state, the signal at point B will also be a sinusoid with the frequency ω_0.

Very intuitive idea: It seems reasonable that an oscillation can be maintained if the signal at B has the same amplitude and phase as the injected signal!

Critical point: -1



(a) Closed loop system



(b) Open loop system

- Tracing signals around the loop, we find that the signals at A and B are identical if there is a frequency ω_0 such that

$$L(i\omega_0) = -1. \tag{1}$$

- This provides a condition for maintaining an **oscillation**.
- The condition (1) implies that the frequency response goes through the value -1, which is called the **critical point**.

Letting ω_c represent a frequency at which $\angle L(i\omega_c) = 180^\circ$,

- we can further reason that the system is stable if $|L(i\omega_c)| < 1$, since the signal at point B will have smaller amplitude than the injected signal.
- A rigorous version is the **Nyquist's stability criterion**.

Outline

Stability: The Routh–Hurwitz Criterion

Loop Transfer Function

Nyquist plot and The Nyquist Criterion

Nyquist[1] plot

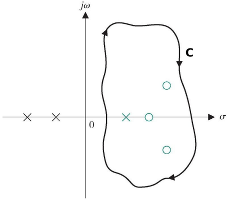
- Frequency response of an LTI system: **Bode plot** of its transfer function
- Stability of a closed-loop system: **Nyquist plot** of its loop transfer function

H. Nyquist (1889 – 1976)

Definition (Nyquist plot)

The **Nyquist plot** of the loop transfer function $L(s)$ is the image of $L(s)$ by tracing $s \in \mathbb{C}$ around the **Nyquist contour**.

- A **contour** is a piecewise smooth path in the complex plane
- A contour is **closed** if it starts and ends at the same point
- A contour is **simple** if it does not cross itself at any point



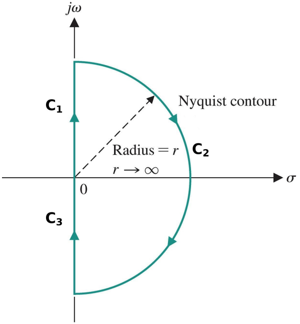
Nyquist's stability criterion utilizes **contours** in the complex plane to relate the **locations** of the open-loop and closed-loop poles.

[1] Harry Nyquist; https://en.wikipedia.org/wiki/Harry_Nyquist

Nyquist contour

The (standard or simplest) Nyquist contour, also known as "Nyquist D contour" ($\Gamma \subset \mathbb{C}$), is made up of three parts:

- **Contour** C_1: points $s = i\omega$ on the positive imaginary axis, as ω ranges from 0 to ∞
- **Contour** C_2: points $s = Re^{i\theta}$ on a semi-circle as $R \to \infty$ and θ ranges from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$
- **Contour** C_3: points $s = i\omega$ on the negative imaginary axis, as ω ranges from $-\infty$ to 0



The image of $L(s)$ when s traverses Γ gives a closed curve in the complex plane and is referred to as the **Nyquist plot** for $L(s)$.

Example 1: a third-order system

Draw a Nyquist plot for $L(s) = \frac{1}{(s+a)^3}$.

- **Part** C_1: $s = i\omega$ with ω from 0 to ∞

$$L(i0) = \frac{1}{a^3}\angle 0^\circ, \qquad L(i\infty) = 0\angle -270^\circ$$

- for $0 < \omega < \infty$

$$L(i\omega) = \frac{1}{(i\omega + a)^3} = \frac{(a - i\omega)^3}{(a^2 + \omega^2)^3} = \frac{a^3 - 3a\omega^2}{(a^2 + \omega^2)^3} + i\frac{\omega^3 - 3a^2\omega}{(a^2 + \omega^2)^3}$$

- **Part** C_2: $s = Re^{i\theta}$ for $R \to \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$.

$$L(Re^{i\theta}) = \frac{1}{(Re^{i\theta} + a)^3} \to 0$$

- **Part** C_3: $s = i\omega$ with $\omega \in (-\infty, 0)$

$$L(-i\omega) = L(\bar{i}\omega) = \overline{L(i\omega)}$$

which is a *reflection* (complex conjugate) of $L(C_1)$ about the real axis.

Example 1: a third-order system

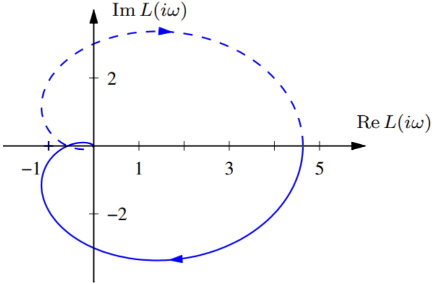
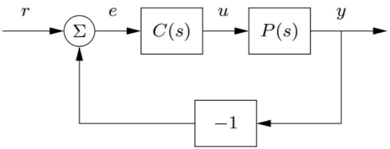
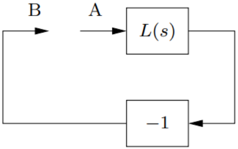


Figure 10.5: Nyquist plot for a third-order transfer function $L(s)$. The Nyquist plot consists of a trace of the loop transfer function $L(s) = 1/(s+a)^3$ with $a = 0.6$. The solid line represents the portion of the transfer function along the positive imaginary axis, and the dashed line the negative imaginary axis. The outer arc of the Nyquist contour Γ maps to the origin.

Simplified Nyquist Criterion



(a) Closed loop system



(b) Open loop system

Theorem (Simplified Nyquist Criterion)

Let $L(s)$ be the loop transfer function for a negative feedback system, and assume that L has no poles in the closed right half-plane ($\mathrm{Re}(s) \geq 0$) except possibly at the origin. Then the closed loop system

$$G_{\mathrm{cl}}(s) = \frac{L(s)}{1 + L(s)}$$

is stable if and only if the image of $L(s)$ along the closed contour Γ has no **net encirclements** *of the critical point $s = -1$.*

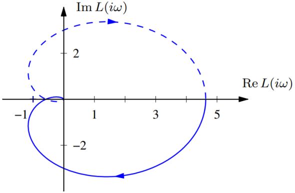
Winding number

The following conceptual procedure can be used to determine that there are no net encirclements.

- Step 1: Fix a pin at the critical point $s = -1$, orthogonal to the plane.
- Step 2: Attach a string with one end at the critical point and the other on the Nyquist plot.
- Step 3: Let the end of the string attached to the Nyquist curve traverse the whole curve.

There are no encirclements if the string does not wind up on the pin when the curve is encircled.

- The number of encirclements is called the **winding number**.



Nyquist plot for $L(s) = \frac{1}{(s+a)^3}$ with $a = 0.6$

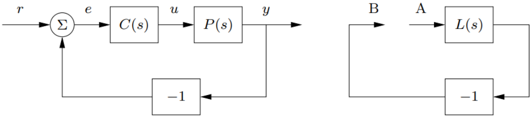
- Closed-loop system

$$G_{\text{cl}}(s) = \frac{L(s)}{1+L(s)} = \frac{1}{(s+0.6)^3+1}, \qquad \lambda_1 = -1.6000, \lambda_{2,3} = -0.1 \pm 0.8660i$$

Summary

- The Routh–Hurwitz Criterion
- The **Loop transfer function**:

$$L(s) = P(s)C(s).$$



(a) Closed loop system (b) Open loop system

- **Nyquist plot** and Simplified **Nyquist criterion**