ECE 171A: Linear Control System Theory
Lecture 17: Nyquist plot and Nyquist Criterion

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Reading materials: Ch 10.2 - 10. 4

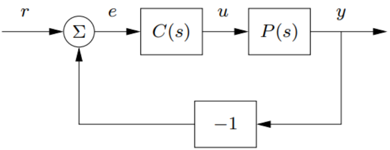
Announcement

- **Midterm exam II**: May 22 (Wednesday in class, Week 8)

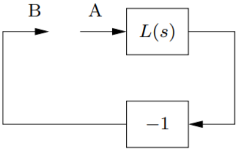
4	Apr 22	L10: Input/output system response (I)	Ch 6.1, 6.2	Practice 1 [Solutions]
	Apr 24	L11: Input/output system response (II)	Ch 6.3	
	Apr 24	D4: Review, HW1/HW2, Two excercies		
	Apr 26	Midterm I - in class		Homework 4 [Solution 4]
5	Apr 29	L12: Transfer function (I)	Ch 9.1, 9.2	Midterm 1 [Solutions]
	May 01	L13: Transfer function (II)	Ch 9.2, 9.3, 9.4	
	May 01	D5: Review on complex numbers		
	May 03	L14: Poles, zeros and Bode plot	Ch 9.5, Ch 9.6	Homework 5
6	May 06	L15: Bode plot	Ch 2.2, Ch 9.6	
	May 08	L16: Routh-Hurwitz stability and Loop transfer functions	Ch 2.2, Ch 10.1	
	May 08	D6: Bode plot examples		
	May 10	L17: Nyquist plot and Nyquist criterion	Ch 10.2 10.3	Homework 6
7	May 13	L18: Stability margins and Root locus	Ch 10.3, Ch 12.5	
	May 15	L19: PID control (I)	Ch 11.1, 11.2	
	May 15	D7: Nyquist plot examples		
	May 17	L20: PID control (II)	Ch 11.2, 11.3	Homework 7
8	May 20	L21: Review		
	May 22	Midterm II - in class		
	May 22	D8: Q&A		
	May 24	L22: Performance specification	Ch 12.1, 12.2	
9	May 27	Memorial Day observance (no lecture)		
	May 29	L23: Loop shaping	Ch 12.3	

- **Office hours**: Week 6 - Week 10
  - Yang Zheng (Tuesdays, 6:30 pm - 8:30 pm; FAH 3002)
  - Chih-fan Pai (Thursdays, 6:30 pm - 8:30 pm; FAH 3002)
  - Brady Liu (Fridays, 6:30 pm - 8:30 pm; FAH 3002)

Nyquist's idea and Critical point -1



(a) Closed loop system



(b) Open loop system

- Nyquist's idea was to first investigate conditions under which oscillations can occur in a feedback loop.
- Tracing signals around the loop, we find that the signals at A and B are identical if there is a frequency ω_0 such that $\rightarrow$ **oscillation**

$$L(i\omega_0) = -1. \tag{1}$$

- The condition (1) implies that the frequency response goes through the value -1, which is called the **critical point**.

Letting ω_c represent a frequency at which $\angle L(i\omega_c) = 180^\circ$,

- we can further reason that the closed-loop system is stable if $|L(i\omega_c)| < 1$.
- A rigorous version is the **Nyquist's stability criterion**.

Outline

Nyquist plot

Nyquist Stability Criterion

Summary

Outline

Nyquist plot

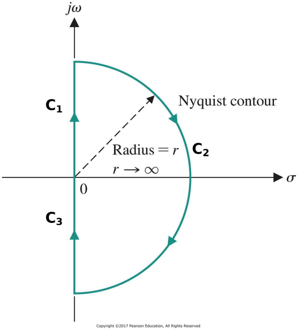
Nyquist Stability Criterion

Summary

Nyquist contour

The (standard or simplest) Nyquist contour, also known as "Nyquist D contour" ($\Gamma \subset \mathbb{C}$), is made up of three parts:

- **Contour** C_1: points $s = i\omega$ on the positive imaginary axis, as ω ranges from 0 to ∞
- **Contour** C_2: points $s = Re^{i\theta}$ on a semi-circle as $R \to \infty$ and θ ranges from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$
- **Contour** C_3: points $s = i\omega$ on the negative imaginary axis, as ω ranges from $-\infty$ to 0



The image of $L(s)$ when s traverses Γ gives a closed curve in the complex plane and is referred to as the **Nyquist plot** for $L(s)$.

Example 1: a third-order system

Draw a Nyquist plot for $L(s) = \frac{1}{(s+a)^3}$.

- **Part** C_1: $s = i\omega$ with ω from 0 to ∞

$$L(i0) = \frac{1}{a^3}\angle 0^\circ, \qquad L(i\infty) = 0\angle - 270^\circ$$

- for $0 < \omega < \infty$

$$L(i\omega) = \frac{1}{(i\omega + a)^3}$$

- **Part** C_2: $s = Re^{i\theta}$ for $R \to \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$.

$$L(Re^{i\theta}) = \frac{1}{(Re^{i\theta} + a)^3} \to 0$$

- **Part** C_3: $s = i\omega$ with $\omega \in (-\infty, 0)$

$$L(-i\omega) = L(\bar{i}\omega) = \overline{L(i\omega)}$$

which is a *reflection* (complex conjugate) of $L(C_1)$ about the real axis.

Example 1: a third-order system

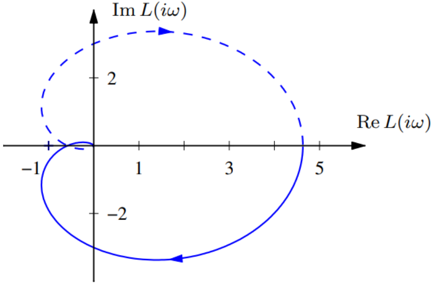


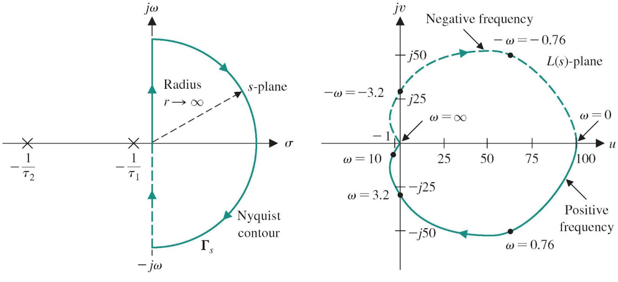
Figure 10.5: Nyquist plot for a third-order transfer function $L(s)$. The Nyquist plot consists of a trace of the loop transfer function $L(s) = 1/(s+a)^3$ with $a = 0.6$. The solid line represents the portion of the transfer function along the positive imaginary axis, and the dashed line the negative imaginary axis. The outer arc of the Nyquist contour Γ maps to the origin.

Example 2: a second-order system

Draw a Nyquist plot for

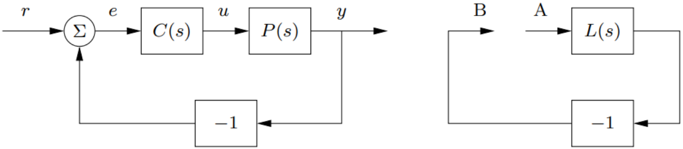
$$L(s) = \frac{100}{(1+s)(1+s/10)}.$$

- **Contour** C_1: $L(i0) = 100\angle 0°$, $L(i\infty) = 0\angle{-180°}$
- **Contour** C_2: $\lim_{R\to\infty} L(Re^{i\theta}) = 0$



(a) (b)

Simplified Nyquist Criterion



(a) Closed loop system (b) Open loop system

Theorem (Simplified Nyquist Criterion)

Let $L(s)$ be the loop transfer function for a negative feedback system, and assume that L has no poles in the closed right half-plane ($\mathrm{Re}(s) \geq 0$) except possibly at the origin ($s = 0$). Then the closed loop system

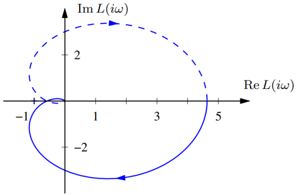
$$G_{\mathrm{cl}}(s) = \frac{L(s)}{1 + L(s)}$$

is stable if and only if the image of $L(s)$ along the closed contour Γ (i.e., its Nyquist plot) has no net encirclements of the critical point $s = -1$.

Winding number

A conceptual procedure to determine that there are no net encirclements.

- Step 1: Fix a pin at the critical point $s = -1$, orthogonal to the plane.
- Step 2: Attach a string with one end at the critical point and the other on the Nyquist plot.
- Step 3: Let the end of the string attached to the Nyquist curve traverse the whole curve.



Nyquist plot for $L(s) = \frac{1}{(s+a)^3}$ with $a = 0.6$

There are no encirclements if the string does not wind up on the pin when the curve is encircled.

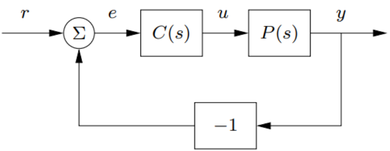
- The number of encirclements is called the **winding number**.

- By the Nyquist stability criterion, the closed-loop system should be stable:

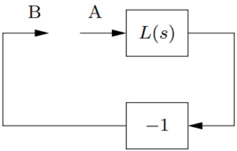
$$G_{\text{cl}}(s) = \frac{L(s)}{1 + L(s)} = \frac{1}{(s+0.6)^3 + 1}, \qquad \lambda_1 = -1.6000, \lambda_{2,3} = -0.1 \pm 0.8660i$$

Outline

Nyquist Stability Criterion



(a) Closed loop system



(b) Open loop system

- Consider the closed-loop transfer function $G_{cl}(s) = \dfrac{L(s)}{1 + L(s)} = \dfrac{L(s)}{\Delta(s)}$
- The **poles** of $\Delta(s)$ are the poles of $L(s)$ — **open-loop poles**
- The **zeros** of $\Delta(s)$ are the poles of $G_{cl}(s)$ — **closed-loop poles**

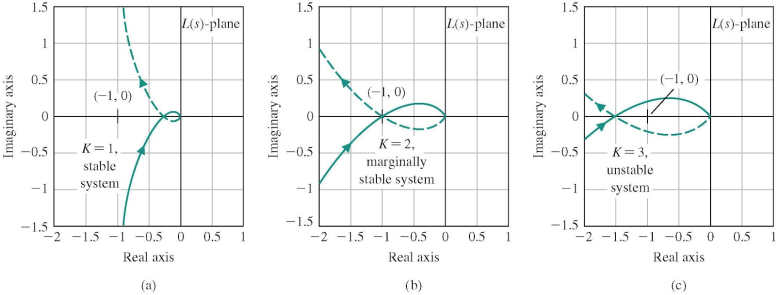
Theorem (Nyquist Stability Criterion)

Consider a unity feedback control system with open-loop transfer function $L(s)$. Let Γ be a Nyquist contour. The closed-loop system is stable if and only if **the number of counterclockwise encirclements** *of the critical point $-1 + i0$ by the Nyquist plot $L(\Gamma)$ is equal to* **the number of open-loop unstable poles** *of $L(s)$.*

Nyquist Stability: Example

Determine the closed-loop stability of the loop transfer function

$$L(s) = \frac{\kappa}{s(1+\tau_1 s)(1+\tau_2 s)} = \frac{\kappa}{s(1+s)^2}$$



- The Nyquist plot crosses the critical point $-1 + i0$ when $\kappa = 2$
- The closed-loop system is stable when $0 < \kappa < 2$.

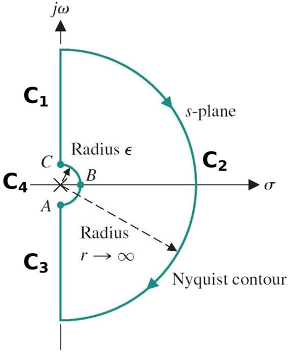
Pole/Zero on the Imaginary Axis

- When the loop transfer function has poles on the imaginary axis, the gain is infinite at the poles.
- The Nyquist contour needs to be modified to take a small detour around such poles or zeros
- So, we add another part: **Contour** C_4
  - plot $L(\epsilon e^{i\theta})$ for $\epsilon \to 0$ and

$$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

  - substitute $s = \epsilon e^{i\theta}$ into $L(s)$ and examine what happens as

$$\epsilon \to 0$$

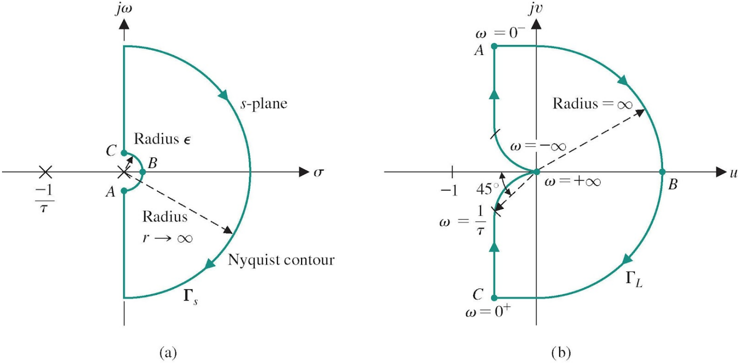


Example 3

Draw a Nyquist plot for a loop transfer system:

$$L(s) = \frac{\kappa}{s(1 + \tau s)}$$

- Since there is a pole at the origin, we need to use a modified Nyquist contour



Example 3

- **Contour** C_4 with $s = \epsilon e^{i\theta}$ for $\epsilon \to 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$:

$$\lim_{\epsilon \to 0} L(\epsilon e^{i\theta}) = \lim_{\epsilon \to 0} \frac{\kappa}{\epsilon e^{i\theta}} = \lim_{\epsilon \to 0} \frac{\kappa}{\epsilon} e^{-i\theta} = \infty \angle -\theta$$

  - The phase of $L(s)$ changes from $\dfrac{\pi}{2}$ at $\omega = 0^-$ to $-\dfrac{\pi}{2}$ at $\omega = 0^+$

- **Contour** C_1 with $\omega \in (0, \infty)$:

$$\begin{aligned} L(i0^+) &= \infty \angle -90^\circ \\ L(i\infty) &= \lim_{\omega \to \infty} \frac{\kappa}{i\omega(1 + i\omega\tau)} = \lim_{\omega \to \infty} \left| \frac{\kappa}{\tau \omega^2} \right| \angle -\pi/2 - \tan^{-1}(\omega\tau) \\ &= 0 \angle -180^\circ \end{aligned}$$

- **Contour** C_2 with $s = re^{i\theta}$ for $r \to \infty$ and θ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$:

$$\lim_{r \to \infty} L(re^{i\theta}) = \lim_{r \to \infty} \left| \frac{\kappa}{\tau r^2} \right| e^{-2i\theta} = 0 \angle -2\theta$$

  - The phase of $L(s)$ changes from $-\pi$ at $\omega = \infty$ to π at $\omega = -\infty$

- **Contour** C_3 with $\omega \in (-\infty, 0)$:
  - $L(C_3)$ is a **reflection** of $L(C_1)$ about the real axis

Proof via Principle of the Argument

- **Principle of the Argument** applied to $\Delta(s) = 1 + L(s)$:
  - Let Γ be a Nyquist contour.
  - Z: the number of zeros of $\Delta(s)$ inside Γ (closed-loop unstable poles).
  - P: the number of poles of $\Delta(s)$ inside Γ (open-loop unstable poles).
  - Then, the image of Γ under $\Delta(s)$, denoted as $\Delta(\Gamma)$, encircles the origin in clockwise direction $N = Z - P$ times.
- Thus, the number of closed-loop poles in the closed right half-plane is:

$$Z = N + P$$

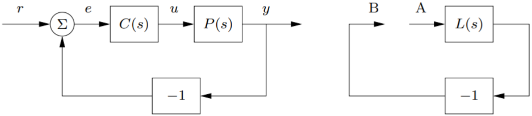
  - N: the **clockwise encirclements of the origin** by $\Delta(\Gamma)$, which corresponds to the **clockwise encirclements of** $-1 + i0$ by the Nyquist plot $L(\Gamma)$
  - P: the number of poles of $\Delta(s)$ inside Γ, which corresponds to the number of **open-loop unstable poles** of $L(s)$.
- The closed-loop stability requires $Z = 0$, thus we need to have $N = -P$:
  - **the number of counterclockwise encirclements** of $-1 + i0$ by the Nyquist plot is equal to **the number of open-loop unstable poles** of $L(s)$.

Outline

Summary

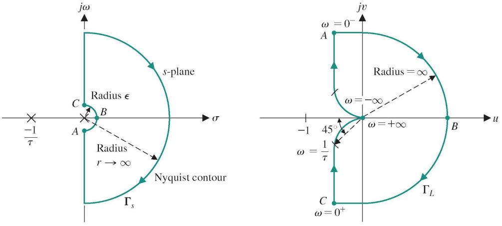
- Nyquist's idea was to first investigate conditions under which oscillations can occur in a feedback loop – The **Loop transfer function**:

$$L(s) = P(s)C(s).$$



(a) Closed loop system (b) Open loop system

- **Nyquist plot** and (Simplified) **Nyquist criterion**



(a) (b)

Summary

Theorem (Simplified Nyquist Criterion)

Let $L(s)$ be the loop transfer function for a negative feedback system, and assume that L has no poles in the closed right half-plane ($\mathrm{Re}(s) \geq 0$) except possibly at the origin. Then the closed-loop system

$$G_{\mathrm{cl}}(s) = \frac{L(s)}{1 + L(s)}$$

is stable if and only if the image of $L(s)$ along the closed contour Γ (i.e., its Nyquist plot) has no net encirclements of the critical point $s = -1$.

Theorem (Nyquist Stability Criterion)

Consider a unity feedback control system with open-loop transfer function $L(s)$. Let Γ be a Nyquist contour. The closed-loop system is stable if and only if **the number of counterclockwise encirclements** *of the critical point $-1 + i0$ by the Nyquist plot $L(\Gamma)$ is equal to* **the number of open-loop unstable poles** *of $L(s)$.*

Outline

Cauchy's Principle of the Argument (for your reference only)

Loop Transfer Function

Loop transfer function

$$L(s) = P(s)C(s) \qquad \Rightarrow \qquad G_{\text{cl}}(s) = \frac{L(s)}{1 + L(s)}.$$

- Consider a control system with a loop transfer function:

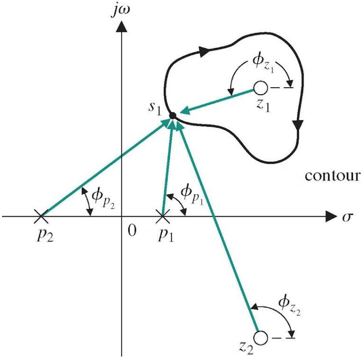
$$L(s) = \kappa \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}$$

- At each s, $L(s)$ is a complex number with magnitude and phase:

$$|L(s)| = |\kappa| \frac{\prod_{i=1}^{m} |s - z_i|}{\prod_{i=1}^{n} |s - p_i|} \qquad \angle L(s) = \angle \kappa + \sum_{i=1}^{m} \angle (s - z_i) - \sum_{i=1}^{n} \angle (s - p_i)$$

- Graphical evaluation of the magnitude and phase:
  - $|s - z_i|$ is the length of the vector from z_i to s
  - $|s - p_i|$ is the length of the vector from p_i to s
  - $\angle (s - z_i)$ is the angle from the real axis to the vector from z_i to s
  - $\angle (s - p_i)$ is the angle from the real axis to the vector from p_i to s

Evaluating $L(s)$ along a Contour



- A zero z_i outside the contour C: The net change in $\angle(s - z_i)$ is 0
- A zero z_i inside the contour C: The net change in $\angle(s - z_i)$ is -2π
- A pole p_i outside the contour C: the net change in $\angle(s - p_i)$ is 0
- A pole p_i inside the contour C: the net change in $\angle(s - p_i)$ is -2π

Evaluating $L(s)$ along a Contour

Let C be a simple closed clockwise contour in the complex plane; Evaluating $L(s)$ at all points on C produces a new closed contour $L(C)$

— **image of C under $L(s)$**.

Assumption: C does not pass through the origin or any of the poles or zeros of $L(s)$ (otherwise $\angle L(s)$ is undefined). **Effects of poles and zeros**:

- A zero z_i outside the contour C:
  - As s moves around the contour C, the vector $s - z_i$ swings up and down but not all the way around
  - Thus, the net change in $\angle(s - z_i)$ is 0
- A zero z_i inside the contour C:
  - As s moves around the contour C, the vector $s - z_i$ turns all the way around
  - Thus, the net change in $\angle(s - z_i)$ is -2π
- A pole p_i outside the contour C: the net change in $\angle(s - p_i)$ is 0
- A pole p_i inside the contour C: the net change in $\angle(s - p_i)$ is -2π

Principle of the Argument

- Let Z and P be the number of zeros and poles of $L(s)$ inside C
- As s moves around C, $\angle L(s)$ undergoes a net change of $-(Z-P)2\pi$
- A net change of -2π means that the vector from 0 to $L(s)$ swings **clockwise** around the origin one full rotation
- A net change of $-(Z-P)2\pi$ means that the vector from 0 to $L(s)$ must encircle the origin in **clockwise** direction $(Z-P)$ times

Theorem (Cauchy's Principle of the Argument)

Consider a transfer function $L(s)$ and a simple closed clockwise contour C. Let Z and P be the number of zeros and poles of $L(s)$ inside C.

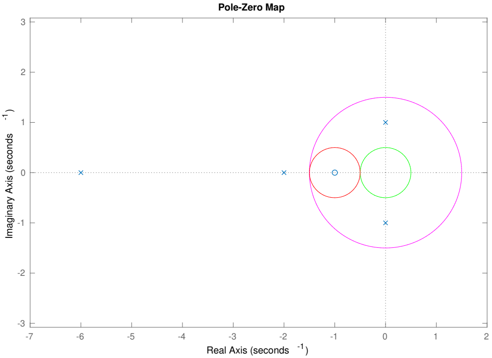
- *Then, the contour generated by evaluating $L(s)$ along C will encircle the origin in a clockwise direction $Z-P$ times.*

- Note that Cauchy's Principle of the Argument works for **any transfer function** — $L(s)$ above does not need to be a loop transfer function.

Principle of the Argument: Example

- Pole-zero map for

$$G(s) = \frac{10(s+1)}{(s+2)(s^2+1)(s+6)}$$



Principle of the Argument: Example

- A circle contour C centered at the origin with radius 0.5 (green)
- The contour may be parameterized by $z(t) = 0.5e^{-it}$ for $t \in [0, 2\pi]$
- The contour C is mapped by $G(s)$ to a new contour (from blue to red), e.g., parameterized by $G(z(t))$ for $t \in [0, 2\pi]$

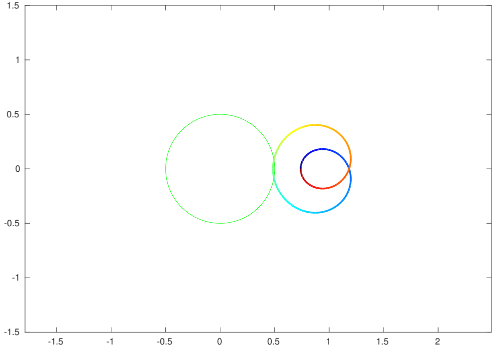


Figure: Encircle the origin in a clockwise direction $Z - P = 0$ times

Principle of the Argument: Example

- A circle contour C centered at $(-1, 0)$ with radius 1 (red)
- The contour C is mapped by $G(s)$ to a new contour (from blue to red)

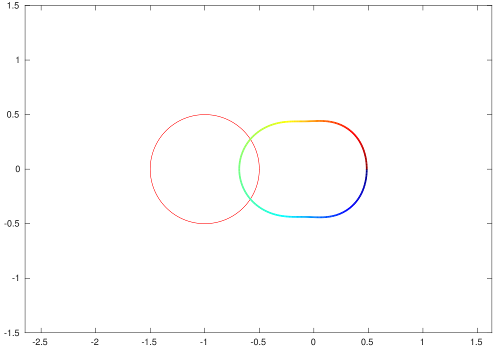


Figure: Encircle the origin in a clockwise direction $Z - P = 1$ time

Principle of the Argument: Example

- A circle contour C centered at the origin with radius 1.5 (magenta)
- The contour C is mapped by $G(s)$ to a new contour (from blue to red)

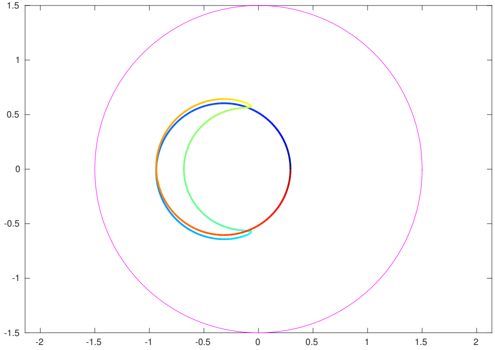


Figure: Encircle the origin in a clockwise direction $Z - P = 1 - 2 = -1$ time