ECE 171A: Linear Control System Theory
Lecture 18: Stability Margins and Root locus

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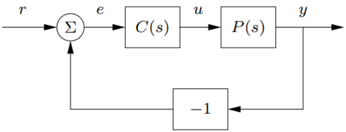
Reading materials: Ch 10.3, Ch 12.5

Outline

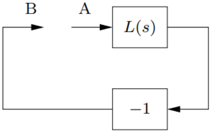
Nyquist's Stability Criterion

- Nyquist's idea was to use the property of the **Loop transfer function** (i.e., Nyquist plot) to determine the closed-loop stability.

$$L(s) = P(s)C(s).$$



(a) Closed loop system



(b) Open loop system

Theorem (Nyquist's Stability Criterion)

Consider a unity feedback control system with open-loop transfer function $L(s)$. Let Γ be a Nyquist contour. The closed-loop system is stable if and only if **the number of counterclockwise encirclements** *of $-1 + i0$ by the Nyquist plot $L(\Gamma)$ is equal to the number of poles of $L(s)$ inside Γ (i.e.* **open-loop unstable poles***).*

Outline

Stability margins

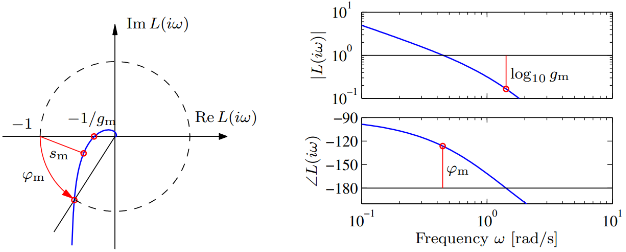
Root locus

Summary

Stability Margin

In practice, it is not enough that a system is stable. There must also be some margins of stability that describe how far from instability the system is and its **robustness to perturbation**.

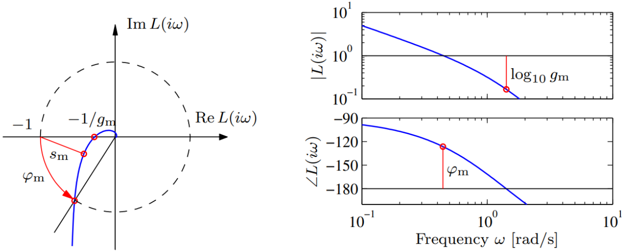
- **Stability margins** express how well the Nyquist curve of the loop transfer avoids the critical point -1.



(a) Nyquist plot (b) Bode plot

- The shortest distance s_m of the Nyquist curve to the critical point is a natural criterion — **stability margin**.

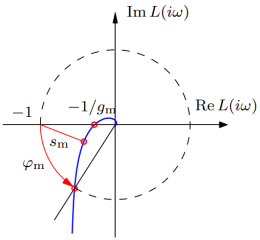
Gain Margin



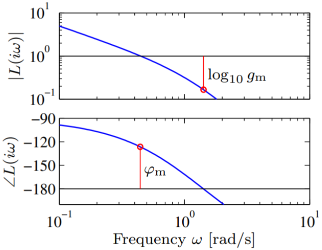
(a) Nyquist plot (b) Bode plot

- **Gain Margin**:
  - the factor by which the open-loop gain can be increased before a stable closed-loop system becomes unstable
  - It is the **inverse** of the distance between the origin and the point between -1 and 0 where the loop transfer function crosses the negative real axis.
  - On a Nyquist plot, the gain margin is the **inverse** of the distance to the first point where $L(s)$ crosses the real axis.

Phase Margin



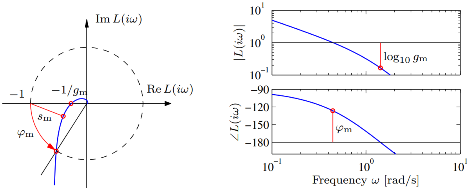
(a) Nyquist plot



(b) Bode plot

- **Phase Margin**:
  - the amount by which the open-loop phase can be decreased before a stable closed-loop system becomes unstable
  - i.e. the amount of phase lag required to reach the stability limit
- On a Nyquist plot, the phase margin is the smallest angle on the unit circle between -1 and $L(s)$

Algebraic Definitions



(a) Nyquist plot (b) Bode plot

- **Phase-Crossover frequency**
  - ω_{pc} at which $L(i\omega)$ crosses the real axis: $\angle L(i\omega_{\mathrm{pc}}) = -180^\circ$
- **Gain Margin**
  - the inverse of the open-loop gain at ω_{pc}: $g_{\mathrm{m}} = \frac{1}{|L(i\omega_{\mathrm{pc}})|}$
- **Gain-Crossover frequency**
  - ω_{gc} at which $G(j\omega)$ crosses the unit circle: $|L(i\omega_{\mathrm{gc}})| = 1$
- **Phase Margin**
  - the amount by which the open-loop phase at ω_g exceeds -180°:

$$\varphi_{\mathrm{m}} = \angle L(i\omega_{\mathrm{gc}}) + 180^\circ$$

Stability margins for a third-order system

Example

Consider a loop transfer function $L(s) = \dfrac{3}{(s+1)^3}$

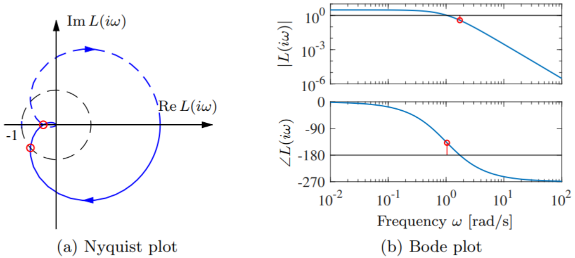


Figure: Stability margins for a third-order transfer function. (a) Nyquist plot; (b) Bode plot.

- We can use its Nyquist plot or Bode plot. This yields the following values:

$$g_{\mathrm{m}} = 2.67, \qquad \varphi_{\mathrm{m}} = 41.7^\circ, \qquad s_{\mathrm{m}} = 0.464.$$

Outline

Root locus - Overview

Motivation: System responses are affected by the locations of the poles of its transfer function in the complex domain, e.g., *stability*, *convergence speed*, etc.

- Feedback control can move the closed-loop system poles by designing an appropriate controller – **pole placement** (not covered in this course).

What is the root locus method? — Another graphical tool

- The **root locus** is a graph of the roots of the characteristic polynomial as a function of a parameter — give insight into the effects of the parameter.
- i.e., the **root locus** provides all possible pole locations as a system parameter (e.g., the controller gain) varies
- **Obtain the root locus** — find the roots of the closed loop characteristic polynomial for different values of the parameter (*easy for computers*).
- The general shape of the root locus can be obtained with very **little computational effort**, and that it often gives *considerable insight*.

Root locus: Example 1

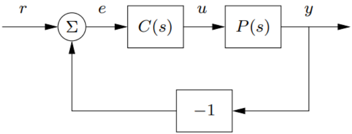


Figure: Feedback control system

- Consider a single-loop feedback control system with

$$P(s) = \frac{1}{s(s+2)}, \qquad C(s) = k$$

- The closed-loop transfer function from the reference r to output y is:

$$G_{\mathrm{yr}}(s) = \frac{kP(s)}{1+kP(s)} = \frac{k}{s^2+2s+k}$$

- How do the closed-loop poles vary as a function of k?
  - We can actually compute the roots as $\lambda_{1,2} = -1 \pm \sqrt{1-k}$.

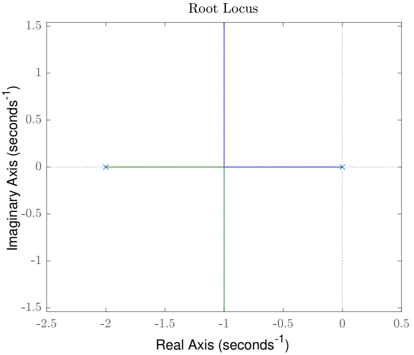
Root Locus: Example 1

Example

- Root locus for

$$P(s) = \frac{1}{s(s+2)}$$

- Matlab command: `rlocus(tf([1],[1 2 0]))`.



Root Locus: Example 2

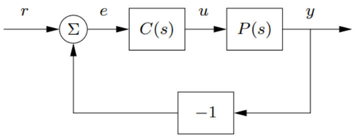


Figure: Feedback control system

Example

- Consider a single-loop feedback control system with

$$P(s) = \frac{(s+3)}{s(s+2)}, \qquad C(s) = k$$

- The closed-loop transfer function from r to y is:

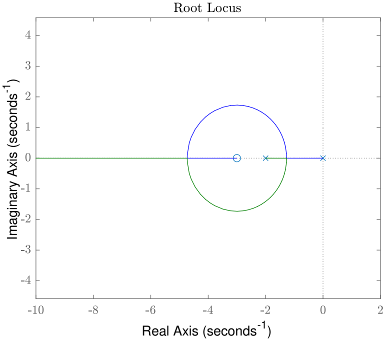
$$G_{\text{yr}}(s) = \frac{kP(s)}{1+kP(s)} = \frac{k(s+3)}{s^2+(2+k)s+3k}$$

Root Locus: Example 2

- Root locus for

$$P(s) = \frac{(s+3)}{s(s+2)}$$

- Matlab command: `rlocus(tf([1 3],[1 2 0]))`.



- In this case, adding a stable zero in the open-loop system increases the relative stability of the closed-loop system by attracting the branches of the root locus.

Root Locus: Example 3

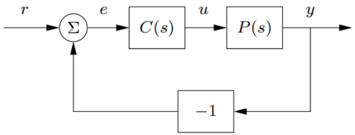


Figure: Feedback control system

Example

- Consider a single-loop feedback control system with

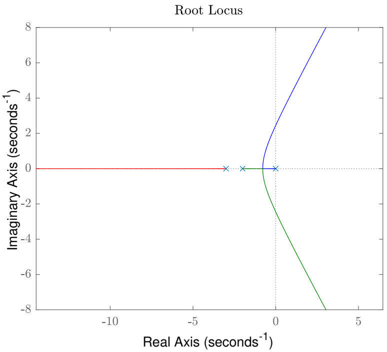
$$P(s) = \frac{1}{s(s+2)(s+3)}, \qquad C(s) = k$$

- The closed-loop transfer function from r to y is:

$$G_{\mathrm{yr}}(s) = \frac{k}{s^3 + 5s^2 + 6s + k}$$

Root Locus: Example 3

- Root locus for
$$P(s) = \frac{1}{s(s+2)(s+3)}$$
- Matlab command: `rlocus(tf([1],[1 5 6 0]))`.



- In this case, adding a stable pole in the open-loop system makes the closed-loop system less stable (stable for some values of k);

Root Locus: Definition

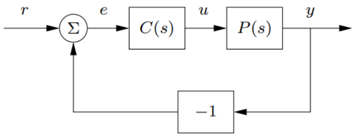


Figure: Feedback control system

- Closed-loop transfer function:

$$G_{\mathrm{yr}}(s) = \frac{kP(s)}{1 + kP(s)}$$

- The closed-loop poles satisfy:

$$1 + kP(s) = 0$$

- The **root locus** is the set of points s such that $1 + kP(s) = 0$ as k varies

Root Locus: Definition

Consider the zeros and poles of $P(s)$ explicitly:

$$P(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$
$$= b_m \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}$$

- The closed loop characteristic polynomial is:

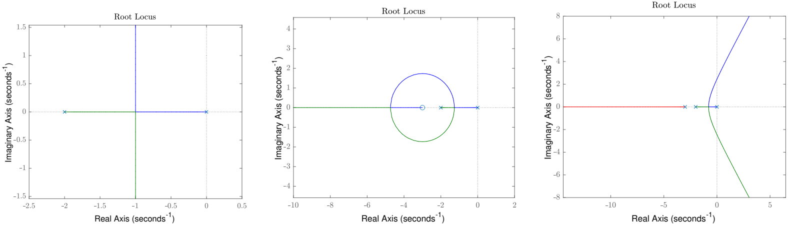
$$1 + kP(s) = 0 \qquad \Rightarrow \qquad a_{\text{cl}}(s) := a(s) + kb(s) = 0$$

- The closed loop poles are the roots of $a_{\text{cl}}(s)$.
- The **root locus** is a graph of the roots of $a_{\text{cl}}(s)$ as the gain k is varied from 0 to ∞.
- Since the polynomial $a_{\text{cl}}(s)$ has degree n, the plot will have n branches.

Starting and ending points of Root locus

- Each branch starts at a different open-loop pole.
- m of the branches end at different open-loop zeros.
- The remaining $n - m$ branches go to infinity

Example



(a) $P(s) = \frac{1}{s(s+2)}$ (b) $P(s) = \frac{s+3}{s(s+2)}$ (c) $P(s) = \frac{1}{s(s+2)(s+3)}$

Starting and ending points of Root locus

- The closed loop characteristic polynomial is:

$$1 + kP(s) = 0 \qquad \Rightarrow \qquad a_{\text{cl}}(s) := a(s) + kb(s) = 0$$

- The **root locus** is a graph of the roots of $a_{\text{cl}}(s)$ as the gain k is varied from 0 to ∞.

Starting points when $k = 0$: we have $a_{\text{cl}}(s) := a(s) + kb(s) = a(s)$.

- The closed-loop poles are equal to the open-loop poles.
- Open-loop poles at $s = p$ with multiplicity $l \Rightarrow$

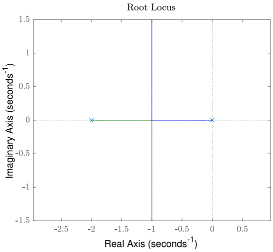
$$a(s) + kb(s) = (s-p)^l \tilde{a}(s) + kb(s) \approx (s-p)^l \tilde{a}(p) + kb(p) = 0$$

For **small value** of k, we have the roots are

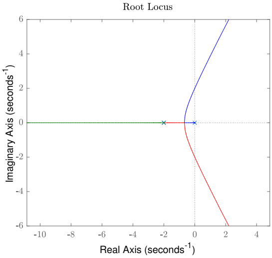
$$s = p + \sqrt[l]{-kb(p)/\tilde{a}(p)}$$

- The root locus has a **star pattern** with l branches from the open-loop pole $s = p$, and the angle between two neighboring branches is $\frac{2\pi}{l}$.

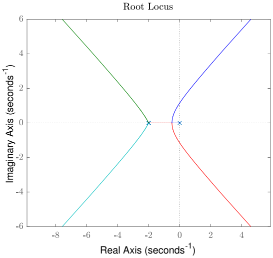
Examples



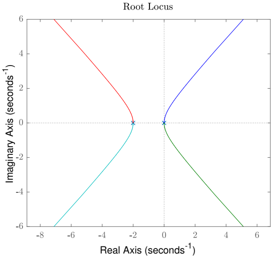
(a) $P(s) = \frac{1}{s(s+2)}$



(b) $P(s) = \frac{1}{s(s+2)^2}$



(c) $P(s) = \frac{1}{s(s+2)^3}$



(d) $P(s) = \frac{1}{s^2(s+2)^2}$

Starting and ending points of Root locus

- The closed loop characteristic polynomial is:

$$1 + kP(s) = 0 \qquad \Rightarrow \qquad a_{\text{cl}}(s) := a(s) + kb(s) = 0$$

- The **root locus** is a graph of the roots of $a_{\text{cl}}(s)$ as the gain k is varied from 0 to ∞.

Ending points when k goes to infinity: we have

$$a_{\text{cl}}(s) := b(s)\left(\frac{a(s)}{b(s)} + k\right) \approx b(s)\left(\frac{s^{n-m}}{b_0} + k\right)$$

- For large K, the **closed-loop poles** are approximately the **roots (zeros of $P(s)$)** of $b(s)$ and

$$\sqrt[n-m]{-b_0 k}$$

- A better approximation of the **closed-loop poles** is

$$s = s_0 + \sqrt[n-m]{-b_0 k}, \quad s_0 = \frac{1}{n-m}\left(\sum_{k=1}^{n} p_k - \sum_{k=1}^{m} z_k\right).$$

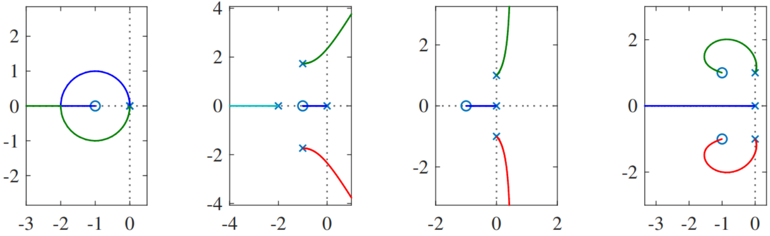
Examples

Example

Show the root loci for the following open-loop transfer functions

$$P_a(s) = \frac{s+1}{s^2}, \qquad P_b(s) = \frac{s+1}{s(s+2)(s^2+2s+4)},$$

$$P_c(s) = \frac{s+1}{s(s^2+1)}, \quad P_d(s) = \frac{s^2+2s+2}{s(s^2+1)}.$$

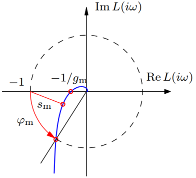


(a) $P_{\mathrm{a}}(s)$ (b) $P_{\mathrm{b}}(s)$ (c) $P_{\mathrm{c}}(s)$ (d) $P_{\mathrm{d}}(s)$

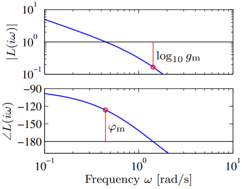
Outline

Summary

- **Stability margins** express how well the Nyquist curve of the loop transfer avoids the critical point -1.
- The shortest distance s_m of the Nyquist curve to the critical point is a natural criterion — **stability margin**; Another two criteria are **gain margin** and **phase margin**.



(a) Nyquist plot



(b) Bode plot

- **Root locus**: a graph of the closed-loop roots as k is varied from 0 to ∞.
  - The plot of root locus will have n branches.
  - Each branch starts at a different open-loop pole.
  - m of the branches end at different open-loop zeros.
  - The remaining $n - m$ branches go to infinity.