ECE 171A: Linear Control System Theory
Lecture 21: Review (L11 - L20)

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Announcements

- Midterm exam (II) in class 9:00 am - 9:50 am, May 22 (this Wednesday)
  - **Scope**: Lectures 11 - 21, HW4 - HW6, HW7 (Q1, Q2), DI 5-7; (Reading materials in the textbook)
  - Closed book, closed notes, closed external links.
  - **Come on time** (1 or 2 minutes early if you can; we will start at 9:00 am promptly)
  - No MATLAB is required. No graphing calculators are permitted. A basic arithmetic calculator is allowed.
  - The exams must be done in a blue book. Bring a blue book with you.
  - **No collaboration and discussions are allowed**. It is dishonest to cheat on exams. Instances of academic dishonesty will be referred to the Office of Student Conduct for adjudication. *You don't want to take a risk for such a small thing*.
- Discussion 8 will be extra office hours (1:00 pm - 1:50 pm).

Outline

Review: L11 - L20

Examples: Nyquist plot

Outline

Review: L11 - L20

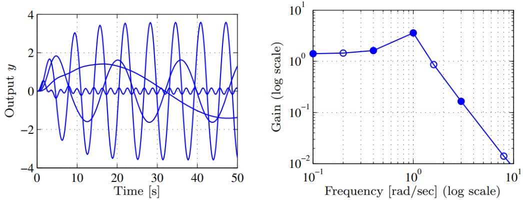
Examples: Nyquist plot

L11 - Input/output responses (II)

- **Impulse response**

$$h(t) = \int_0^t Ce^{A(t-\tau)}B\delta(\tau)d\tau + D\delta(t) = Ce^{At}B + D\delta(t).$$

- **Frequency responses**



(a) Time domain simulations (b) Frequency response

- **The convolution equation** (You don't need to memorize this equation)

$$y(t) = \underbrace{Ce^{At}x(0)}_{\text{Initial response}} + \underbrace{\int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)}_{\text{forced response}}.$$

L12: Transfer function (I)

- Transient response and steady-state response to an exponential input e^{st}

$$y(t) = \underbrace{Ce^{At}\left(x(0) - (sI - A)^{-1}B\right)}_{\text{transient}} + \underbrace{\left(C(sI - A)^{-1}B + D\right)e^{st}}_{\text{steady-state}}$$

- Transfer function

$$G(s) = C(sI - A)^{-1}B + D.$$

  - Express the steady-state solution of a **stable** linear system forced by a sinusoidal input

$$u(t) = \sin(\omega t) \quad \rightarrow \quad y_{\mathrm{ss}} = |G(i\omega)| \sin(\omega t + \angle G(i\omega))$$

- **Frequency domain modeling**: Modeling a system through its response to sinusoidal and exponential signals.
  - The **transfer function** provides a complete representation of a linear system in the frequency domain.
  - We represent the dynamics of the system in terms of the generalized frequency s rather than the time domain variable t.

Example

Consider a system

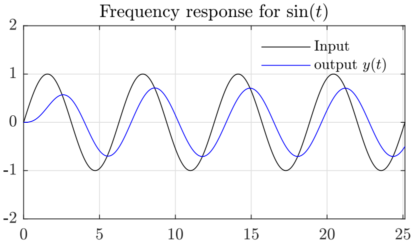
$$G(s) = \frac{1}{s^2 + s + 2}.$$

- For a stable system, the steady-state response to input $u(t) = \sin \omega t$ is

$$y = M \sin(\omega t + \theta), \quad \text{where} M = |G(i\omega)|, \theta = \arg(G(i\omega))$$

- **Suppose**: $u(t) = \sin t$. What is the steady state of the output $y(t)$?

$$G(i\omega) = \frac{1}{(i\omega)^2 + i\omega + 2}, \qquad M = |G(i)| = \frac{1}{\sqrt{2}}, \quad \theta = -45^\circ$$



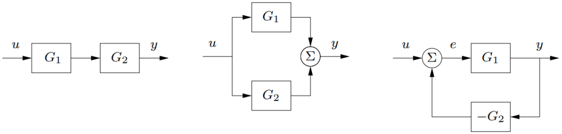
L13: Transfer function (II)

- Transfer function for linear ODEs

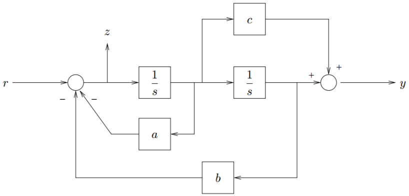
$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \ldots + b_0 u,$$

$$G(s) = \frac{b_m s^m + b_{m-1}s^{n-1} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}.$$

- Block diagram with transfer functions



(a) $G_{yu} = G_2 G_1$ (b) $G_{yu} = G_1 + G_2$ (c) $G_{yu} = \frac{G_1}{1 + G_1 G_2}$



(Problem 3 in HW5)

Common transfer functions

Type	System	Transfer function
Integrator	$\dot{y} = u$	$\frac{1}{s}$
Differentiator	$y = \dot{u}$	s
First-order system	$\dot{y} + ay = u$	$\frac{1}{s+a}$
Double integrator	$\ddot{y} = u$	$\frac{1}{s^2}$
Damped oscillator	$\ddot{y} + 2\zeta\omega_0\dot{y} + \omega_0^2 y = u$	$\frac{1}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$
State-space system	$\dot{x} = Ax + Bu$ $y = Cx + Du$	$C(sI - A)^{-1}B + D$
PID controller	$y = k_\mathrm{p}u + k_\mathrm{d}\dot{u} + k_\mathrm{i}\int u$	$k_\mathrm{p} + k_\mathrm{d}s + \frac{k_\mathrm{i}}{s}$
Time delay	$y(t) = u(t-\tau)$	$e^{-\tau s}$

Table: Transfer functions for some common linear time-invariant systems.

Example: calculating transfer function

Consider an LTI system

$$\dot{x}_1 = -a_1 x_1 - a_2 x_2 + u \qquad\qquad y = x_2$$
$$\dot{x}_2 = x_1$$

- **Method 1**: The system matrices are

$$A = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}, D = 0.$$

- Compute its transfer function

$$\begin{aligned} G(s) = C(sI - A)^{-1}B + D &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s + a_1 & a_2 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{s^2 + a_1 s + a_2} \begin{bmatrix} s & -a_2 \\ 1 & s + a_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{s^2 + a_1 s + a_2}. \end{aligned}$$

- **Method 2**: Compute its transfer function directly from ODEs.

L14: Zeros, Poles and Bode plot

- The **features** of a transfer function are often associated with **important system properties**.
  - zero frequency gain: the steady-state value of a step response for a **stable** system
  - the locations of the poles and zeros: **Poles** — stability of a system; **Zeros** – Block transmission of certain signals

Poles (eigenvalues) of the matrix A = Poles of the transfer function $G(s)$

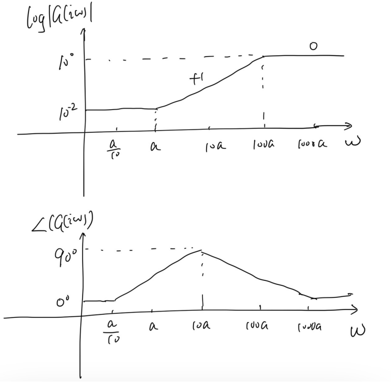
- The **Bode plot** gives a quick overview of a stable linear system $G(s)$
- Its frequency response $G(i\omega)$ can be represented by two curves — **Bode plot**
  - **Gain curve**: gives $|G(i\omega)|$ as a function of frequency ω — log/log scale (traditionally in dB — $20 \log |G(i\omega)|$; we often consider $\log |G(i\omega)|$)
  - **Phase curve**: gives $\angle G(i\omega)$ as a function of frequency ω — log/linear scale in degrees

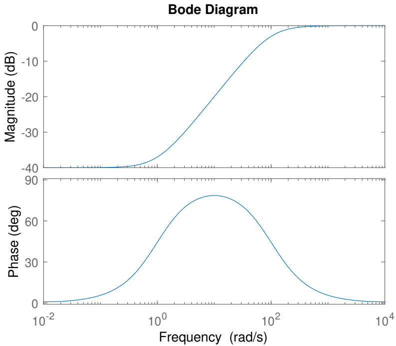
L15: Bode plot

Draw a Bode plot for $G_2(s) = \dfrac{s+a}{s+100a}$

- Step 1: find breakpoints (related to poles and zeros): a, 100a.
- Step 2: Calculate $|G(i0)|$ and $\angle G(i0)$ to determine the starting points
- Step 3: Sketch the bode plot by the rules
  - **Magnitude increases with a zero**: if the zero is a first-order real zero, the slop is $+1$; if the zero is a second-order zero (or complex zero), the slop is $+2$
  - **Magnitude decreases with a pole**: If there pole is a first-order real pole, the slop is -1; if the pole is a second-order pole (or complex pole), the slop is -2
  - **Phases changes** by $+90$ with a first order real zero; $+180$ with a second order zero (or complex zero). The change starts around $a/10$ and ends around $10a$.
  - **Phases changes** by -90 with a first order real pole; -180 with a second order pole (or complex pole). Similarly, the change starts around $a/10$ and ends around $10a$.

Example 1: bode plot



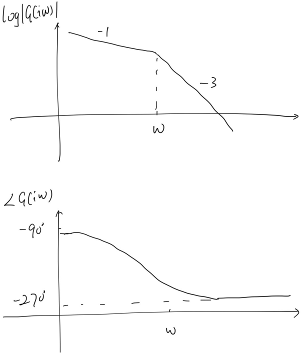


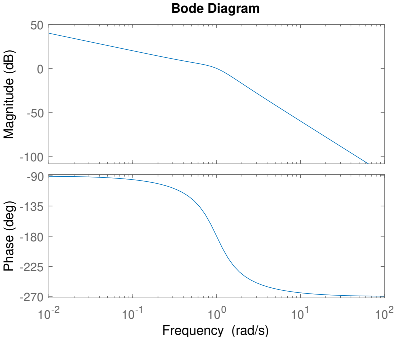
- The breakpoint frequencies occur at $\omega = a$ and $\omega = 100a$ ($s = -a$ is a zero and $s = -100a$ is a pole).
- The magnitude curve starts with $\log|G(i0)| = -2$ and a slop of 0, and the slop increases by 1 at $\omega = a$, then decreases by -1 at $\omega = 100a$.
- The phase curve starts with 0°, transients from 0° to 90°, and then transients from 90° to 0°. The phase transient is from $a/10$ to $10a$

Example 2: bode plot

Consider a transfer function

$$G_4(s) = \frac{1}{s(s^2 + 2\zeta\omega_0 s + \omega_0^2)}$$





- The breakpoint frequencies of this system are 0 and ω_0.
- The gain curve starts with a slop of -1 and the slop decreases by -2 at ω_0.
- The phase curve starts with $-90°$, transients from $-90°$ to $-270°$, with a period from $\omega_0/10$ to $10\omega_0$.

L15: Routh-Hurwitz stability

Theorem

Consider a Routh table from the polynomial $a(s)$ in

$$G(s) = \frac{b(s)}{a(s)}.$$

- *The number of sign changes in the first column of the Routh table is equal to the number of roots of $a(s)$ in the closed right half-plane.*

Corollary (BIBO Stability of LTI Systems)

The system $G(s)$ is **BIBO stable** *if and only if there are no sign changes in the first column of its Routh table.*

- You don't need to memorize the general Routh table. We will give it to you if needed.

Example: Higher-order System

Example

Consider the characteristic polynomial of a fifth-order system:

$$a(s) = s^5 + s^4 + 10s^3 + 72s^2 + 152s + 240$$

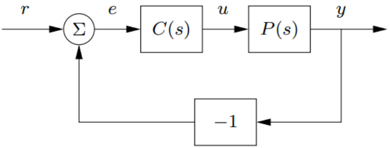
- The Routh table is:

s^5	1	10	152
s^4	1	72	240
s^3	-62	-88	0
s^2	70.6	240	0
s^1	122.6	0	0
s^0	240	0	0

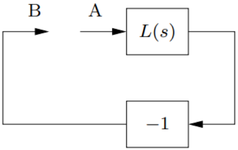
- Since there are two sign changes in the first column, there are two roots in the right half-plane and the system is **unstable**
- The roots of $a(s)$ are:

$$a(s) = (s+3)(s+1 \pm j\sqrt{3})(s-2 \pm j4)$$

Stability of feedback systems



(a) Closed loop system



(b) Open loop system

- **Lyapunov stability** — eigenvalue test of the closed-loop matrix; e.g.,

$$\begin{aligned} \text{Dynamics} \rightarrow \quad & \dot{x} = Ax + Bu, \\ \text{Feedback controller} \rightarrow \quad & u = -Kx \end{aligned} \quad \Rightarrow \quad \dot{x} = (A - BK)x.$$

- **Poles or The Routh–Hurwitz Criterion**;

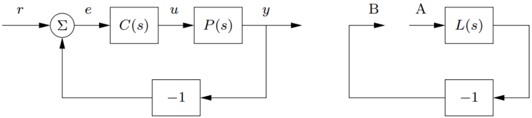
$$\begin{cases} P(s) & = \dfrac{n_{\rm p}(s)}{d_{\rm p}(s)} \\ C(s) & = \dfrac{n_{\rm c}(s)}{d_{\rm c}(s)} \end{cases} \quad \Rightarrow \quad G_{yr}(s) = \frac{PC}{1+PC} = \frac{n_{\rm p}(s) n_{\rm c}(s)}{d_{\rm p}(s) d_{\rm c}(s) + n_{\rm p}(s) n_{\rm c}(s)}$$

They are **straightforward but give little guidance** for design: it is not easy to tell how the controller should be modified to make an unstable system stable.

L16: Loop transfer functions and Nyquist plot

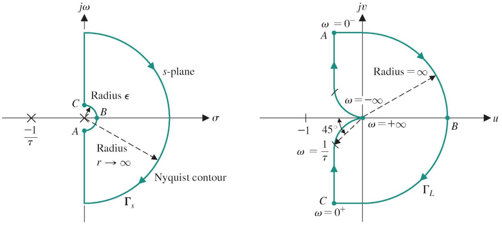
- Nyquist's idea was to first investigate conditions under which oscillations can occur in a feedback loop – The **Loop transfer function**:

$$L(s) = P(s)C(s).$$



(a) Closed loop system (b) Open loop system

- **Nyquist plot** and Simplified **Nyquist criterion**



(a) (b)

L17: Nyquist plot and Nyquist Criterion

Theorem (Nyquist Stability Criterion)

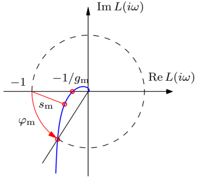
Consider a negative feedback control system with open-loop transfer function $L(s)$. Let Γ be a Nyquist contour.

- *The closed-loop system is stable if and only if* **the number of counterclockwise encirclements** *of the critical point $-1 + i0$ by the Nyquist plot $L(\Gamma)$ is equal to* **the number of open-loop unstable poles** *of $L(s)$.*

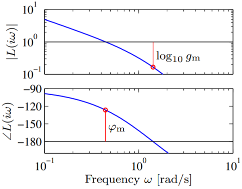
- Another version of Nyquist stability criterion:
  - $1 + L(s)$ has $Z = N + P$ zeros in the right half plane (i.e., **closed-loop unstable poles**),
  - where P is the number of open-loop unstable poles and N is the number of clockwise encirclements of -1 by the Nyquist plot.

L18 & L19: Stability margins and Root locus

- **Stability margins** express how well the Nyquist curve of the loop transfer avoids the critical point -1.
- The shortest distance s_m of the Nyquist curve to the critical point is a natural criterion — **stability margin**; Another two criteria are **gain margin** and **phase margin**.



(a) Nyquist plot



(b) Bode plot

- **Root locus**: a graph of the closed-loop roots as k is varied from 0 to ∞.

  - The plot of root locus will have n branches.
  - Each branch starts at a different open-loop pole.
  - m of the branches end at different open-loop zeros.
  - The remaining $n - m$ branches go to infinity.

L20: PID control

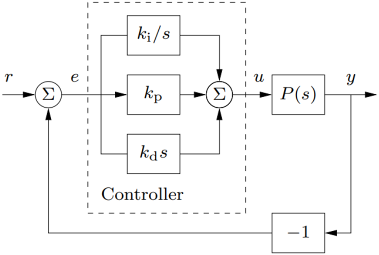


Figure: PID using error feedback

PID control

- the proportional term (P) — the **present** error;
- the integral term (I) — the **past** errors;
- the derivative term (D) — anticipated **future** errors.

- **Magic of integral action**

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau) d\tau.$$

$$\Rightarrow \quad u_0 = k_p e_0 + k_i \lim_{t\to\infty} \int_0^t e(\tau) d\tau.$$

- PID controller for lower-order (1st and 2nd order) systems

PID Control Example

Example

Consider the plant

$$P(s) = \frac{1}{s^2 - 3s - 1}$$

Design a PID controller $C(s)$ to achieve step response with zero steady-state error and place the closed-loop system poles at -1, -2, -3

- PID controller: $C(s) = k_\mathrm{p} + \frac{k_\mathrm{i}}{s} + k_\mathrm{d} s$
- Closed-loop transfer function:

$$G_{\mathrm{yr}}(s) = \frac{PC}{1 + PC} = \frac{k_\mathrm{d} s^2 + k_\mathrm{p} s + k_\mathrm{i}}{s^3 + (k_\mathrm{d} - 3)s^2 + (k_\mathrm{p} - 1)s + k_\mathrm{i}}$$

- Matching coefficients with

$$\begin{aligned} p(s) &= (s+1)(s+2)(s+3) \\ &= (s^2 + 3s + 2)(s+3) \\ &= s^3 + 6s^2 + 11s + 6, \end{aligned}$$

  we have $k_\mathrm{d} = 9$, $k_\mathrm{p} = 12$, $k_\mathrm{i} = 6$.

Outline

Review: L11 - L20

Examples: Nyquist plot

Example 4

- Another version of Nyquist stability criterion: $1 + L(s)$ has $Z = N + P$ zeros in the right half plane (i.e., **closed-loop unstable poles**), where P is the number of open-loop unstable poles and N is the number of clockwise encirclements of -1 by the Nyquist plot.

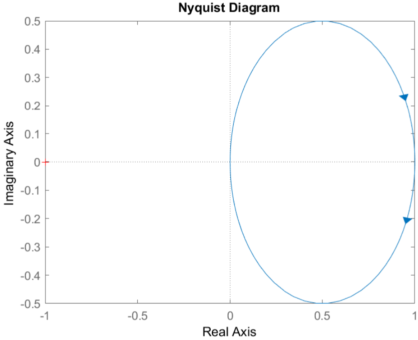


Figure: Nyquist plot for $L(s) = \frac{1}{s+1}$

$$L(s) = \frac{1}{s+1}$$

$$Z = N + P = 0$$

Then,

$$G_{\text{yr}} = \frac{L(s)}{1 + L(s)}$$
$$= \frac{1}{s+2}$$

Example 5

$$L(s) = \frac{1}{(s+1)^2}$$

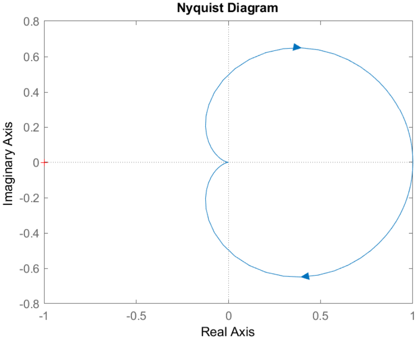


Figure: Nyquist plot for $L(s) = \frac{1}{(s+1)^2}$

$$Z = N + P = 0$$

Then,

$$G_{\mathrm{yr}} = \frac{L(s)}{1 + L(s)} = \frac{1}{s^2 + 2s + 2}$$

Closed-loop poles

$$p_{1,2} = -1 \pm 1i$$

Example 6

$$L(s) = \frac{1}{s(s+1)}$$

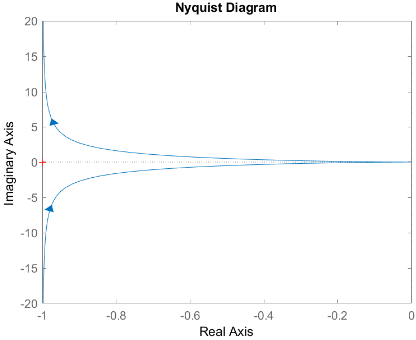


Figure: Nyquist plot for $L(s) = \frac{1}{s(s+1)}$

$$Z = N + P = 0$$

Then,

$$G_{\text{yr}} = \frac{L(s)}{1 + L(s)} = \frac{1}{s^2 + s + 1}$$

Closed-loop poles

$$p_{1,2} = -0.5 \pm 0.866i$$

Example 7

$$L(s) = \frac{1}{s(s+1)(s+0.5)}$$

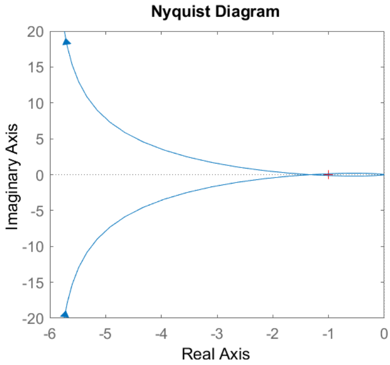


Figure: Nyquist plot for $L(s) = \frac{1}{s(s+1)(s+0.5)}$

$$Z = N + P = 2$$

Then,

$$G_{\text{yr}} = \frac{L(s)}{1 + L(s)}$$
$$= \frac{1}{s^3 + 1.5s^2 + 0.5s + 1}$$

Closed-loop poles

$$p_{1,2} = 0.0416 \pm 0.7937i$$
$$p_3 = -1.5832$$

Midterm II

Good Luck!