

# **ECE 171A: Linear Control System Theory**

## **Lecture 22: Performance specifications**

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# Outline

Sensitivity Functions

Performance specifications

Summary

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## Feedback control systems

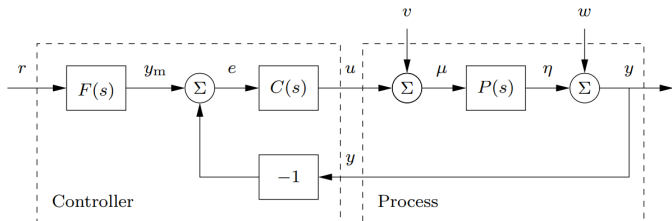


Figure: Block diagram of a control system with two degrees of freedom (DOF).

- ▶ The 2 DOF controller:
  - a feedback block  $C(s)$
  - a feedforward block  $F(s)$ .
- ▶ The external signals
  - the reference signal  $r$ ,
  - the load disturbance  $v$ ,
  - the measurement noise  $w$ .
- ▶ The measured output is  $y$ , and the control signal is  $u$ .

**Loop shaping:** design the behavior of the *closed-loop system* by focusing on the *open-loop transfer function*.

- ▶ **Example — the Nyquist criterion:** we used the Nyquist plot for the open loop transfer function to determine the stability of the closed-loop system

## Closed-loop transfer functions

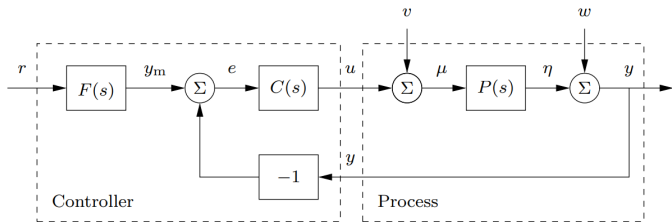


Figure: Block diagram of a control system with two degrees of freedom.

- ▶ The feedback control system is composed of linear elements;
- ▶ The relations between the signals can be expressed via transfer functions.

The overall system has three external inputs: the **reference**  $r$ , the **load disturbance**  $v$ , and the **measurement noise**  $w$ .

- ▶ Any of the remaining signals can be relevant for design, but the most common ones are
  - the error  $e$ , the input  $u$ , and the output  $y$ .

## Closed-loop transfer functions

**Table 12.1:** Transfer functions relating the signals of the control system in Figure 12.1. The external inputs are the reference signal  $r$ , load disturbance  $v$ , and measurement noise  $w$ , represented by each row. The columns represent the measured signal  $y$ , control input  $u$ , error  $e$ , process input  $\mu$ , and process output  $\eta$  that are most relevant for system performance.

$y$	$u$	$e$	$\mu$	$\eta$	
$\frac{PCF}{1+PC}$	$\frac{CF}{1+PC}$	$\frac{F}{1+PC}$	$\frac{CF}{1+PC}$	$\frac{PCF}{1+PC}$	$\left. \begin{array}{l} r \\ v \\ w \end{array} \right\}$
$\frac{P}{1+PC}$	$\frac{-PC}{1+PC}$	$\frac{-P}{1+PC}$	$\frac{1}{1+PC}$	$\frac{P}{1+PC}$	
$\frac{1}{1+PC}$	$\frac{-C}{1+PC}$	$\frac{-1}{1+PC}$	$\frac{-C}{1+PC}$	$\frac{-PC}{1+PC}$	

For most control designs, we focus on the following subset — the **Gang of Six**

$$\begin{aligned}
 G_{yr} &= \frac{PCF}{1+PC}, & -G_{uv} &= \frac{PC}{1+PC}, & G_{yv} &= \frac{P}{1+PC} \\
 G_{ur} &= \frac{CF}{1+PC}, & -G_{uw} &= \frac{C}{1+PC}, & G_{yw} &= \frac{1}{1+PC}
 \end{aligned}$$

## Sensitivity functions: Gang of four

The response of the system to load disturbances  $v$  and measurement noise  $w$  is of particular importance — referred to as **sensitivity functions**.

- ▶ Sensitivity function

$$S = \frac{1}{1 + PC} \quad \rightarrow \quad G_{yw}$$

- ▶ Complementary sensitivity function

$$T = \frac{PC}{1 + PC} \quad \rightarrow \quad -G_{uv}$$

- ▶ Load sensitivity function

$$PS = \frac{P}{1 + PC} \quad \rightarrow \quad G_{yv}$$

- ▶ Noise sensitivity function

$$CS = \frac{C}{1 + PC} \quad \rightarrow \quad -G_{uw}$$

Important in feedback control design — called the **Gang of Four**

- ▶ The remaining two in the **Gang of Six** are

$$G_{yr} = \frac{PCF}{1 + PC},$$

$$G_{ur} = \frac{CF}{1 + PC}$$

- ▶ feedback controller  $C(s)$  → provide good response w.r.t.  $w$  and  $v$ ;
- ▶ feedforward controller  $F(s)$  → obtain good reference tracking w.r.t.  $r$ .

## Sensitivity functions: Gang of four

### Gang of Four

$$\begin{aligned} S &= \frac{1}{1 + PC} & PS &= \frac{P}{1 + PC} \\ T &= \frac{PC}{1 + PC} & CS &= \frac{C}{1 + PC} \end{aligned}$$

Many interesting properties (assuming stability for the reference tracking)

- ▶ It directly follows that

$$S(s) + T(s) = 1, \quad \forall s \in \mathbb{C}$$

- ▶ The loop transfer function  $PC$  will typically go to zero for large  $s$ , which implies that  $T(s) \rightarrow 0$  and  $S(s) \rightarrow 1$  as  $s \rightarrow \infty$ .
- ▶ It will not be possible to track very high-frequency reference signals  $|G_{yr}| = |FT| \rightarrow 0$
- ▶ For controllers with integral action,  $PC$  will typically go to infinity for small  $s$  — **low frequency references are tracked well.**



# Outline

Sensitivity Functions

Performance specifications

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# Performance specifications

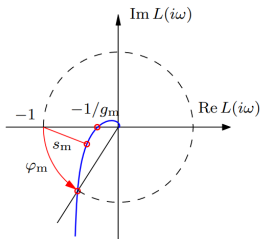
A key element in controller design is how we specify the desired performance.

Specifications capture **robustness** to process variations and **performance** w.r.t.

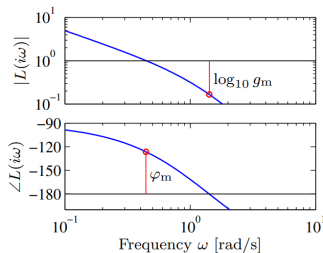
- ▶ the ability to **track** reference signals and **attenuate** load disturbances **without injecting** too much measurement noise.
- ▶ Expressed in terms of transfer functions (the Gang of Six) and the loop transfer function, using features of their **time and frequency responses**.

## Robustness to process variations

- ▶ *Loop transfer function*: Gain/phase margin  $g_m, \varphi_m$ , and stability margin  $s_m$ .



(a) Nyquist plot



(b) Bode plot

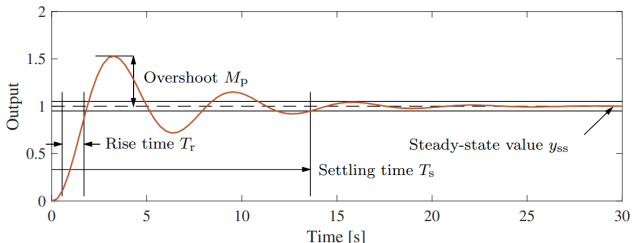
# Performance specifications

Specifications capture **robustness** to process variations and **performance** w.r.t.

- ▶ the ability to **track** reference signals and **attenuate** load disturbances **without injecting** too much measurement noise.
- ▶ Expressed in terms of transfer functions (the Gang of Six) and the loop transfer function, using features of their **time and frequency responses**.

## Features of step responses

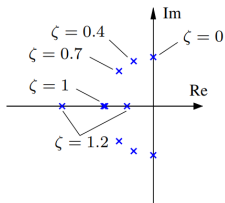
- ▶ Overshoot, rise time, and settling time



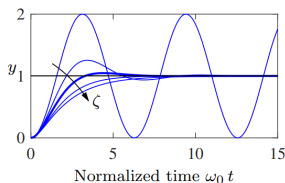
## Time-domain specifications

**Table 7.1:** Properties of the step response for a second-order system  $\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2q = k\omega_0^2u$  with  $0 < \zeta \leq 1$ .

Property	Value	$\zeta = 0.5$	$\zeta = 1/\sqrt{2}$	$\zeta = 1$
Steady-state value	$k$	$k$	$k$	$k$
Rise time (inverse slope)	$T_r = e^{\varphi/\tan\varphi} / \omega_0$	$1.8/\omega_0$	$2.2/\omega_0$	$2.7/\omega_0$
Overshoot	$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$	16%	4%	0%
Settling time (2%)	$T_s \approx 4/\zeta\omega_0$	$8.0/\omega_0$	$5.6/\omega_0$	$4.0/\omega_0$



(a) Eigenvalues

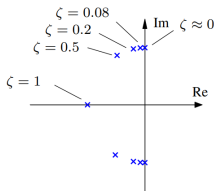


(b) Step responses

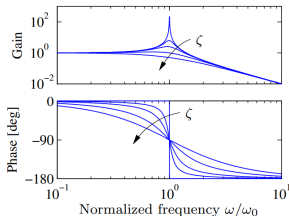
# Frequency-domain specifications

**Table 7.2:** Properties of the frequency response for a second-order system  $\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2q = k\omega_0^2u$  with  $0 < \zeta \leq 1$ .

Property	Value	$\zeta = 0.1$	$\zeta = 0.5$	$\zeta = 1/\sqrt{2}$
Zero frequency gain	$M_0$	$k$	$k$	$k$
Bandwidth	$\omega_b = \omega_0 \sqrt{1 - 2\zeta^2 + \sqrt{(1 - 2\zeta^2)^2 + 1}}$	$1.54 \omega_0$	$1.27 \omega_0$	$\omega_0$
Resonant peak gain	$M_r = \begin{cases} k/(2\zeta\sqrt{1 - \zeta^2}) & \zeta \leq \sqrt{2}/2, \\ \text{N/A} & \zeta > \sqrt{2}/2 \end{cases}$	$5k$	$1.15k$	$k$
Resonant frequency	$\omega_{mr} = \begin{cases} \omega_0\sqrt{1 - 2\zeta^2} & \zeta \leq \sqrt{2}/2, \\ 0 & \zeta > \sqrt{2}/2 \end{cases}$	$\omega_0$	$0.707\omega_0$	$0$



(a) Eigenvalues



(b) Frequency responses

## Response to Reference Signals

The responses of the output  $y$  and the control signal  $u$  to the reference  $r$  are

$$G_{yr} = \frac{PCF}{1 + PC}, \quad G_{ur} = \frac{CF}{1 + PC}.$$

- ▶ Peak (or resonant) value  $M_r$ , the peak frequency  $\omega_{mr}$  the bandwidth  $\omega_r$ .

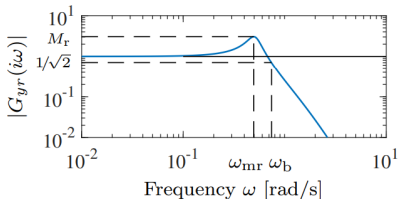


Figure: Gain curve of the transfer function  $G_{yr}$

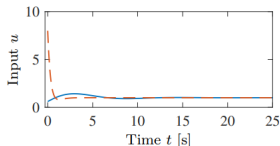
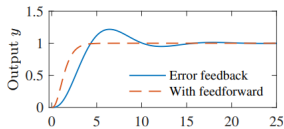
- ▶ We can choose  $F = 1$
- ▶ But in many cases it is useful to retain the ability to shape the input/output response by using  $F(s) \neq 1$  — **the full Gang of Six**

# Example

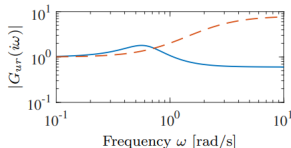
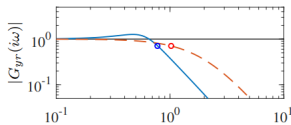
## Example

Consider a process with the transfer function and PI controller

$$P(s) = \frac{1}{(s+1)^3}, \quad C(s) = 0.6 + \frac{0.5}{s}$$



(a) Step responses



(b) Frequency responses

With feedforward  
controller  $F(s)$

- ▶ **Faster response, no overshoot**
- ▶ **Larger bandwidth, no peak**

# Response to Load Disturbances and Measurement Noise

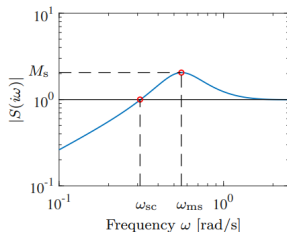
Sensitivity function

$$S = \frac{1}{1 + PC} \rightarrow G_{yw}$$

Load sensitivity function

$$PS = \frac{P}{1 + PC} \rightarrow G_{yv}$$

- ▶ The sensitivity function  $S$  directly shows how feedback  $C(s)$  influences the response of the output to disturbances  $w$
- ▶ Disturbances with frequencies such that  $|S(i\omega)| < 1$  are **attenuated**; such that  $|S(i\omega)| > 1$  are **amplified** by feedback
- ▶ Sensitivity **crossover** frequency  $\omega_{sc}$  is the lowest frequency such that  $|S(i\omega)| = 1$ .



(a) Gain curves



# Response to Load Disturbances and Measurement Noise

**Load sensitivity function**  $PS = \frac{P}{1 + PC} \rightarrow G_{yv}$

- ▶ Load disturbances typically have low frequencies.
- ▶ For strictly proper plant,  $P(s)$  is small for low frequencies.
- ▶ For processes with  $P(0) \neq 0$  and controllers with integral action, we have

$$C(s) \approx \frac{k_i}{s} \Rightarrow G_{yv} \approx \frac{1}{C} = \frac{s}{k_i}$$

for small  $s$  (low frequencies)

A controller with integral action thus **attenuates disturbances with low frequencies** effectively, and the gain  $k_i$  is a measure of disturbance attenuation.

Similar analysis for other transfer functions, e.g., **high-frequency roll-off**

$$-G_{uw} = \frac{C}{1 + PC} \approx C \quad \text{for large } s.$$

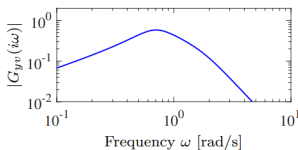
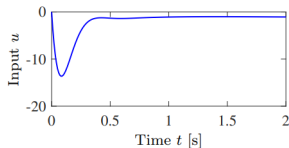
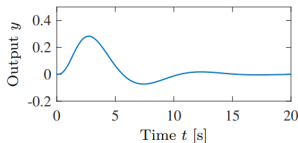
- ▶ It is useful to filter the derivative so that  $C(s)$  go to zero for large  $s$ .

## Example

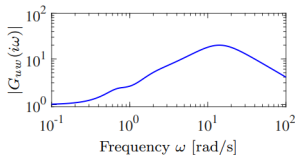
### Example

Consider a process with the transfer function and PID controller

$$P(s) = \frac{1}{(s + 1)^3}, \quad C(s) = \frac{k_d s^2 + k_p s + k_i}{s(s^2 T_f^2 / 2 + s T_f + 1)}$$



(a) Output load response



(b) Input noise response

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## Summary

- ▶ **Sensitivity functions:** for most control designs we focus on the following subset — the **Gang of Six**

$$G_{yr} = \frac{PCF}{1+PC}, \quad -G_{uv} = \frac{PC}{1+PC}, \quad G_{yv} = \frac{P}{1+PC}$$
$$G_{ur} = \frac{CF}{1+PC}, \quad -G_{uw} = \frac{C}{1+PC}, \quad G_{yw} = \frac{1}{1+PC}$$

- ▶ Specifications capture **robustness** to process variations and **performance** w.r.t.
  - the ability to 1) **track** reference signals and 2) **attenuate** load disturbances **without injecting** too much measurement noise.
- ▶ Expressed in terms of transfer functions (e.g., the Gang of Six) and the loop transfer function, using features of their **time and frequency responses**.