ECE 171A: Linear Control System Theory
Lecture 24: Uncertainty and robustness

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Reading materials: Ch 13.1, 13.2

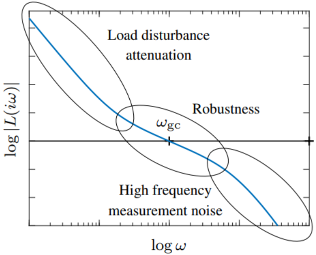
Outline

Outline

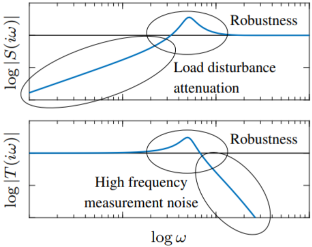
Loop-shaping: Summary

Loop shaping: choose a compensator $C(s)$ that gives a loop transfer function $L(s) = P(s)C(s)$ with a desired shape. — **Trial and error procedure**

- The **loop transfer function** should have roughly the shape below



(a) Gain plot of loop transfer function



(b) Gain plot of sensitivity functions

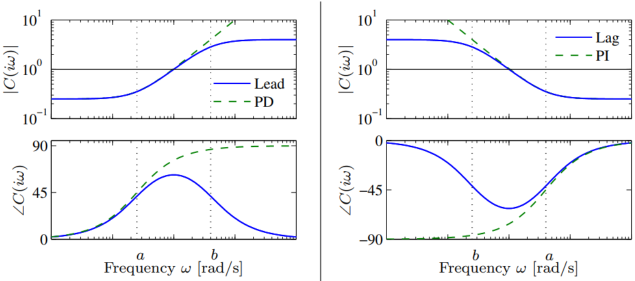
- Good performance requires that the loop transfer function $L(s)$
  - is large for low frequencies — **good tracking of reference signals**
  - has **good attenuation** of low-frequency load disturbances.

Lead and Lag Compensation

Simple compensators with transfer function

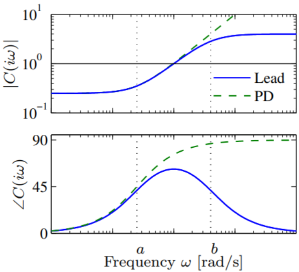
$$C(s) = k\frac{s+a}{s+b}, \qquad a > 0,\ b > 0$$

- **Lag compensator** (offer phase lag) if $a > b$; a PI controller is a special case with $b = 0$.
- **Lead compensator** (offer phase lead) if $a < b$; a PD controller with filtering.

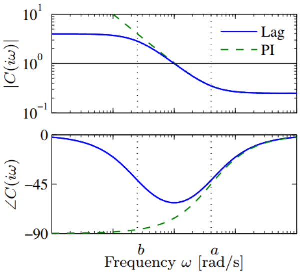


(a) Lead compensation, $a < b$ (b) Lag compensation, $b < a$

Lag and Lead compensations



(a) Lead compensation, $a < b$



(b) Lag compensation, $b < a$

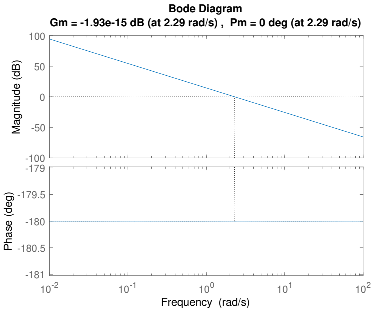
- General purpose of **Lag compenstation**
  - increases the gain at low frequencies
  - improve tracking performance at low frequencies
  - improve disturbance attenuation at low frequencies
- General purpose of **Lead compenstation**
  - Add phase lead in the frequency range between the pole and zero pair
  - By appropriately choosing the location of this phase lead, we can provide **additional phase margin** at the gain crossover frequency.

Example (Example 12.5)

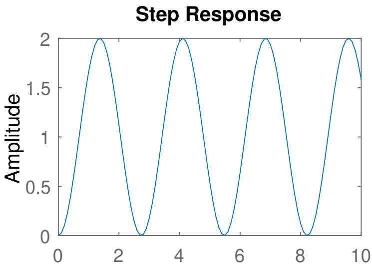
The transfer function for the system dynamics is

$$P(s) = \frac{r}{Js^2}, \qquad r = 0.25, \quad J = 0.0475$$

- less than 1 % error in steady state; $\leq$ 10% tracking error up to 10 rad/s

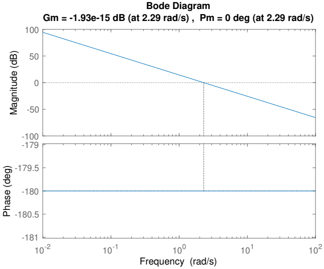


(a) Bode plot with margins



(b) Step response for unit negative feedback

Example 2 - Lead compensation



(a) $P(s)$

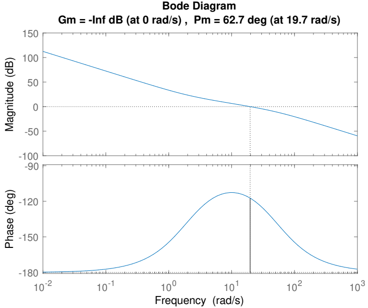
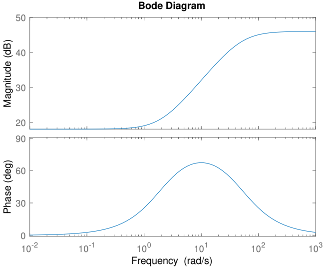


Figure: Margins for $L(s) = P(s)C(s)$



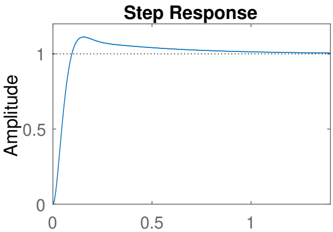
(b) Lead compensation

Consider a lead compensation

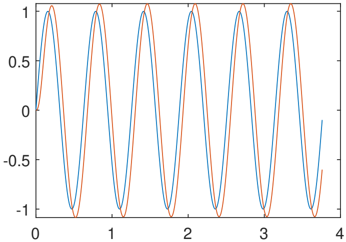
$$C(s) = k\frac{s+a}{s+b},$$

$$a = 2, b = 50, k = 200;$$

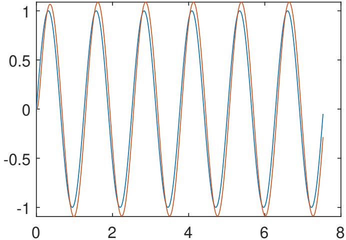
Example 2 - time domain simulations



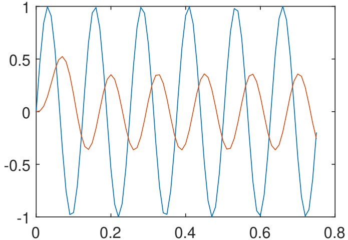
(a) Step response



(b) Frequency response $\omega = 10$



(c) Frequency response $\omega = 5$



(d) Frequency response $\omega = 50$

Outline

Robustness to uncertainty

Robustness to uncertainty is one of the most useful properties of feedback

- This makes it possible to design feedback systems based on *strongly simplified models*.

Recall from **Lecture 3 Fundamental properties of feedback**:

- Disturbance attenuation
- Reference signal tracking
- Robustness to uncertainty
- Shaping of dynamical behavior

Robustness to uncertainty

Robustness to uncertainty is one of the most useful properties of feedback

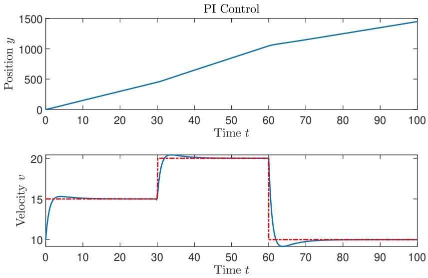
- This makes it possible to design feedback systems based on *strongly simplified models*.

- We consider a simpler scenario, where some system parameters have variations (imprecise measurement).

Cruise control. Condition: $v_0 = 10m/s$, $m = 500\text{kg}, \delta = 0.5$;
PI controller: $K_\text{p} = 250,\ K_\text{i} = 50$

Case 1: Mass change - $m = 200kg$
Flat road ($\theta = 0$)
Piece-wise constant desired velocity signal

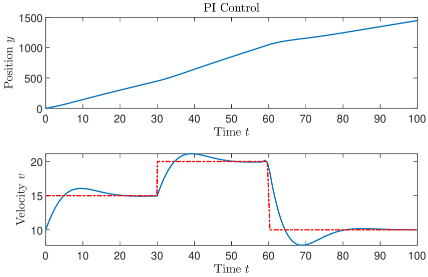
$$v_\text{des} = \begin{cases} 15m/s & t \leq 30 \\ 20m/s & 30 < t \leq 60 \\ 10m/s & 60 < t \end{cases}$$



Robustness to uncertainty

Case 2: Mass change - $m = 800kg$
Flat road ($\theta = 0$)
Piece-wise constant desired velocity signal

$$v_{\text{des}} = \begin{cases} 15m/s & t \leq 30 \\ 20m/s & 30 < t \leq 60 \\ 10m/s & 60 < t \end{cases}$$



Robustness: The same PI controller can make the closed-loop system follow a reference signal even when some system parameters are not known exactly.

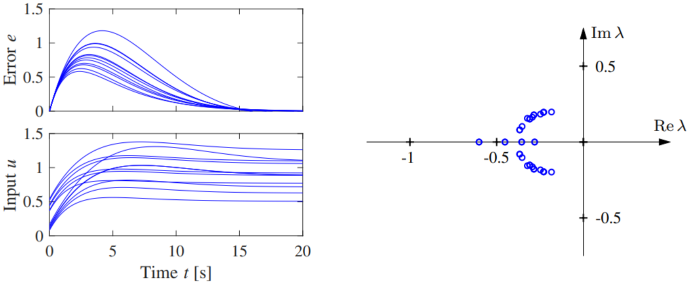
Uncertainty Modeling

We discuss two types of uncertainties in this lecture.

- **Parametric uncertainty** in which the parameters describing the system are not precisely known, e.g.,
  - The variation of the mass of a car, which changes with the number of passengers and the weight of baggage
  - When linearizing a nonlinear system, the parameters of the linearized model also depend on the *operating conditions*.
- **Unmodeled dynamics**, in which some dynamics are neglected during the modeling, e.g.,
  - In Cruise Control, we did not include a detailed model of the engine dynamics

Parametric Uncertainty

- In principle, it is easy to investigate the **effects of parametric uncertainty** by evaluating the performance criteria for a range of parameters.
- Such a calculation reveals the consequences of parameter variations.



(a) Disturbance response

(b) Closed loop eigenvalues

Figure 13.1: Responses of the cruise control system to a slope increase of 4° (a) and the eigenvalues of the closed loop system (b). Model parameters are swept over a wide range. The closed loop system is of second order.

- However, this can be intractable (**computationally demanding**) for large parameter space. *Formal guarantees can be challenging too!*.

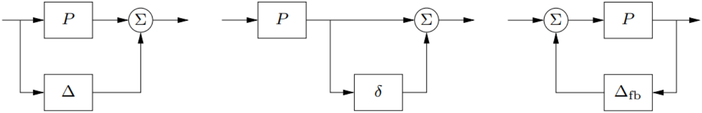
Unmodeled dynamics

How to handle unmodeled dynamics?

- **Method 1:** develop a more complex model that includes additional details.
  - Such models are commonly used for controller development, but substantial effort is required to generate them.
  - These models are themselves likely to be uncertain, since the parameter values may vary over time.

- **Method 2:** investigate whether the closed loop system can be made **insensitive** to generic forms of unmodeled dynamics.
  - The basic idea is to *augment* the nominal model with a *bounded input/output transfer function* that captures the gross features of the unmodeled dynamics.
  - Describing unmodeled dynamics with transfer functions permits us to handle infinite-dimensional systems like time delays.

Unmodeled dynamics



(a) Additive uncertainty (b) Multiplicative uncertainty (c) Feedback uncertainty

Figure 13.2: Unmodeled dynamics in linear systems. Uncertainty can be represented using additive perturbations (a), multiplicative perturbations (b), or feedback perturbations (c). The nominal system is P, and Δ, δ, and Δ_{fb} represent unmodeled dynamics.

- **Additive uncertainty**: the true plant dynamics are in the range of

$$\tilde{P}(s) = P(s) + \Delta(s), \qquad |\Delta(i\omega)| < \epsilon, \qquad \forall \omega \in \mathbb{R}.$$

- **Multiplicative uncertainty**:

$$\tilde{P}(s) = P(s)(1 + \delta(s)), \qquad |\delta(i\omega)| < \epsilon, \qquad \forall \omega \in \mathbb{R}.$$

- **Feedback uncertainty**: $\tilde{P}(s) = \dfrac{P}{1 + P\Delta_{\text{fb}}}, \qquad |\Delta_{\text{fb}}(i\omega)| < \epsilon, \forall \omega \in \mathbb{R}$
- The specific form that is used depends on what provides the best representation of the unmodeled dynamics.

When Are Two Systems Similar?

- A naive approach is to say that two systems are close
  - if their open loop responses are close.
  - or if their open loop frequency responses are similar.
- Unfortunately, both are **inappropriate**!
- This seemingly innocent problem is not as simple as it may appear
- Proper measures are relatively recent (1990s) — **Vinnicombe metric** (details are beyond the scope of this class)

Example

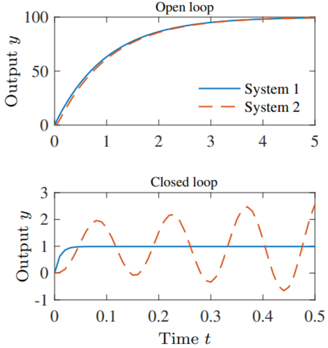
Systems similar in open loop but different in closed loop

$$P_1(s) = \frac{k}{s+1},$$

$$P_2(s) = \frac{k}{(s+1)(sT+1)^2},$$

have very similar open-loop step responses for small values of T.

- Closed loop step responses are different.



(a) Example 13.2

When Are Two Systems Similar?

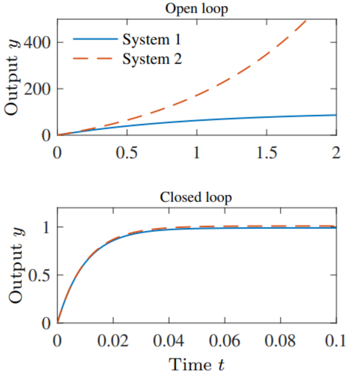
Example

Systems different in open loop but similar in closed loop

$$P_1(s) = \frac{k}{s+1},$$
$$P_2(s) = \frac{k}{s-1},$$

have very different open loop step responses.

- Closed loop step responses are very similar.



(b) Example 13.3

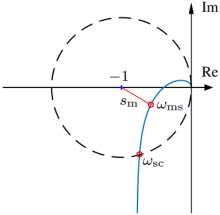
- Two systems can have very close frequency responses (i.e., Bode plots and Nyquist plots are similar)
- But their closed-loop response are very different! (see Example 13.4)
- Proper measures are relatively recent (in the early 90s) — **Vinnicombe metric** (details are not required in this class)

Outline

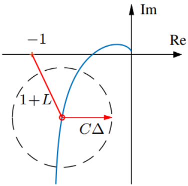
Robust stability

Robust stability: when can we formally show that the stability of a system is robust with respect to process variations?

- **Nyquist criterion**: a powerful and elegant way to study the effects of uncertainty.
- The stability margin s_m is a good robustness measure.



(a) Nyquist plot



(b) Additive uncertainty

Figure 13.7: Illustrations of robust stability in Nyquist plots. The plot (a) shows the stability margin $s_{\mathrm{m}} = 1/M_{\mathrm{s}}$. The plot (b) shows the Nyquist curve and the circle shows uncertainty due to stable additive process variations Δ.

Robust stability - explicit conditions

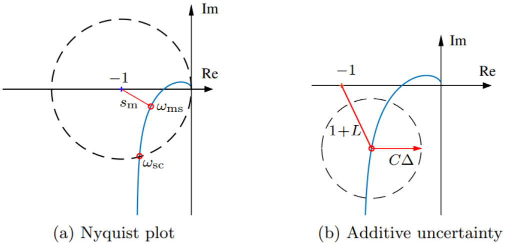


Figure 13.7: Illustrations of robust stability in Nyquist plots. The plot (a) shows the stability margin $s_m = 1/M_s$. The plot (b) shows the Nyquist curve and the circle shows uncertainty due to stable additive process variations Δ.

- If the process is changed from $P(s)$ to $P(s) + \Delta(s)$, the loop transfer function changes from $P(s)C(s)$ to

$$(P(s) + \Delta(s))C(s).$$

- Assume that $\Delta(s)$ is stable, the closed-loop system remains stable as long as the perturbed loop transfer function

$$(P + \Delta)C$$

never reaches the critical point -1.

Robust stability - explicit conditions

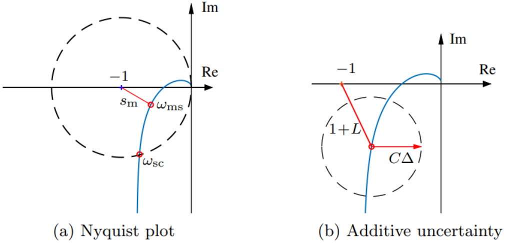


Figure 13.7: Illustrations of robust stability in Nyquist plots. The plot (a) shows the stability margin $s_m = 1/M_s$. The plot (b) shows the Nyquist curve and the circle shows uncertainty due to stable additive process variations Δ.

- The distance from -1 to $L = PC$ is $|1 + L|$.
- The perturbed Nyquist curve will not reach -1 provided that

$$|C\Delta| < |1 + L| \qquad (1)$$

- (1) holds if

$$|\Delta| < \left|\frac{1+L}{C}\right|, \quad \text{or} \quad |\delta| < \left|\frac{1+L}{L}\right| = \frac{1}{|T|}, \quad \text{where } \delta = \frac{\Delta}{P} \qquad (2)$$

Robust stability - explicit conditions

The condition (2) must be valid all all points on the Nyquist curve — point-wise for all frequencies

$$|\delta(i\omega)| < \left| \frac{1 + L(i\omega)}{L(i\omega)} \right| = \frac{1}{|T(i\omega)|}, \quad \forall \omega \geq 0. \tag{3}$$

- **Condition** (3) **is one of the reasons why feedback systems work so well in practice.**
  - The models used to design control systems are often simplified, and the properties of a process may change during operation.
  - Condition (3) implies that the closed loop system will at least be stable for substantial variations in the process dynamics

The peak value of the complimentary sensitivity:

$$M_t = \max_{\omega} |T(i\omega)| := \left\| \frac{PC}{1 + PC} \right\|_{\infty}$$

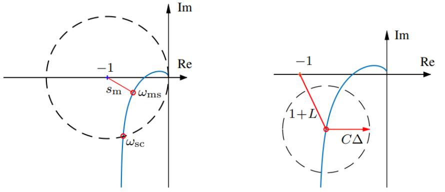
- Condition (3) becomes $|\delta(i\omega)| < 1/M_t, \forall \omega \geq 0.$
- Reasonable values of M_t are from 1.2 to 2.

Outline

Summary

- **Robustness to uncertainty** is one of the most useful properties of feedback — design feedback systems based on *strongly simplified models*.
  - **Parametric uncertainty** in which the parameters describing the system are not precisely known
  - **Unmodeled dynamics**, in which some dynamics are neglected during the modeling.
- An explicit sufficient **robustness condition** based on Nyquist criterion

$$|C\Delta| < |1+L|, \quad \text{or} \quad |\delta(i\omega)| < \left|\frac{1+L(i\omega)}{L(i\omega)}\right| = \frac{1}{|T(i\omega)|}, \quad \forall\omega \geq 0.$$



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