ECE 171A: Linear Control System Theory
Lecture 25: Robust Stability and Fundamental Limits

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Reading materials: Ch 14.1, 14.2

Final Exam

Final Exam — 8:00 am - 10:30 am, June 12

- **Scope**: Lectures 1 - 27, HW1 - HW8, DI 1-10; (Reading materials in the textbook)
- This final exam is closed book but you can bring **one sheet of notes**
  - page maximum size: Letter; can be double-sided;
  - should be hand-written (you can also write on your iPad and print it);
  - it is not acceptable to directly copy-paste lecture notes/HW/textbook.
- The exams must be done in a blue book. Bring a blue book with you.
- No MATLAB is required. No graphing calculators are permitted. You need a basic arithmetic calculator for simple calculations.
- **No collaboration and discussions are allowed**. It is dishonest to cheat on exams. Instances of academic dishonesty will be referred to the Office of Student Conduct for adjudication. *You don't want to take a risk for such a small thing*.

Student Evaluations of Teaching (SET)

- You should have got the following link from UCSD Online Evaluations to evaluate ECE 171A
  `https://academicaffairs.ucsd.edu/Modules/Evals/?e11360527`
- **Deadline**: Saturday, June 08 at 8:00 am
- Your responses are completely anonymous.
- It's your opportunity to let your voices be heard (by our team, the department, and the university).
- Please give some **thoughtful** and **constructive** feedback.
- If you like the course, please say it explicitly and we'd love to hear it
- If you think some aspects can be further improved, we are more than happy to know them (we have implemented some suggestions from previous surveys)

Many thanks for your efforts and your time in this course!

Outline

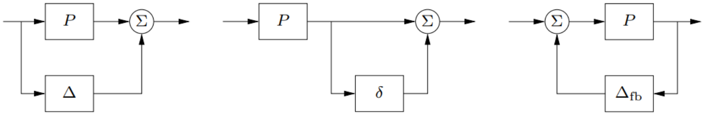
Robust stability

Fundamental limits - System design considerations

Summary

Outline

Unmodeled dynamics



(a) Additive uncertainty (b) Multiplicative uncertainty (c) Feedback uncertainty

Figure 13.2: Unmodeled dynamics in linear systems. Uncertainty can be represented using additive perturbations (a), multiplicative perturbations (b), or feedback perturbations (c). The nominal system is P, and Δ, δ, and Δ_{fb} represent unmodeled dynamics.

- **Additive uncertainty**: the true plant dynamics are in the range of

$$\tilde{P}(s) = P(s) + \Delta(s), \qquad |\Delta(i\omega)| < \epsilon, \qquad \forall \omega \in \mathbb{R}.$$

- **Multiplicative uncertainty**:

$$\tilde{P}(s) = P(s)(1 + \delta(s)), \qquad |\delta(i\omega)| < \epsilon, \qquad \forall \omega \in \mathbb{R}.$$

- **Feedback uncertainty**: $\tilde{P}(s) = \dfrac{P}{1 - P\Delta_{\text{fb}}}, \qquad |\Delta_{\text{fb}}(i\omega)| < \epsilon, \forall \omega \in \mathbb{R}$
- The specific form that is used depends on what provides the best representation of the unmodeled dynamics.

Robust stability

Robust stability: when can we formally show that the stability of a system is robust with respect to process variations?

- **Nyquist criterion**: a powerful and elegant way to study the effects of uncertainty.
- The stability margin s_m is a good robustness measure.

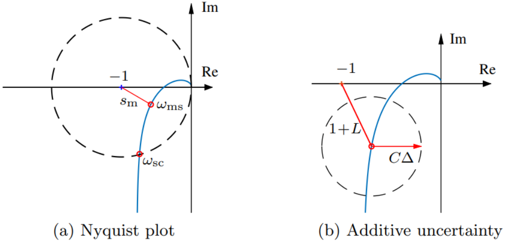
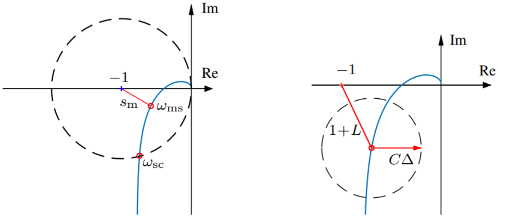


Figure 13.7: Illustrations of robust stability in Nyquist plots. The plot (a) shows the stability margin $s_{\mathrm{m}} = 1/M_{\mathrm{s}}$. The plot (b) shows the Nyquist curve and the circle shows uncertainty due to stable additive process variations Δ.

Robust stability - explicit conditions



(a) Nyquist plot (b) Additive uncertainty

Figure 13.7: Illustrations of robust stability in Nyquist plots. The plot (a) shows the stability margin $s_m = 1/M_s$. The plot (b) shows the Nyquist curve and the circle shows uncertainty due to stable additive process variations Δ.

- If the process is changed from $P(s)$ to $P(s) + \Delta(s)$, the loop transfer function changes from $P(s)C(s)$ to

$$(P(s) + \Delta(s))C(s).$$

- Assume that $\Delta(s)$ is stable, the closed-loop system remains stable as long as the perturbed loop transfer function

$$(P + \Delta)C$$

never reaches the critical point -1.

Robust stability - explicit conditions

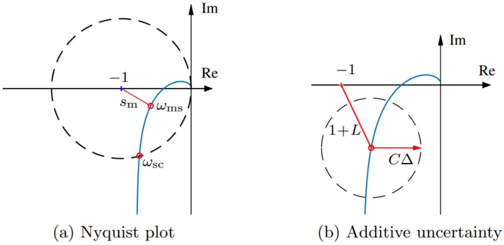


Figure 13.7: Illustrations of robust stability in Nyquist plots. The plot (a) shows the stability margin $s_m = 1/M_s$. The plot (b) shows the Nyquist curve and the circle shows uncertainty due to stable additive process variations Δ.

- The distance from -1 to $L = PC$ is $|1 + L|$.
- The perturbed Nyquist curve will not reach -1 provided that

$$|C\Delta| < |1 + L| \qquad (1)$$

- (1) holds if

$$|\Delta| < \left|\frac{1+L}{C}\right|, \quad \text{or} \quad |\delta| < \left|\frac{1+L}{L}\right| = \frac{1}{|T|}, \quad \text{where} \quad \delta = \frac{\Delta}{P} \qquad (2)$$

Robust stability - explicit conditions

The condition (2) must be valid all all points on the Nyquist curve — point-wise for all frequencies

$$|\delta(i\omega)| < \left|\frac{1 + L(i\omega)}{L(i\omega)}\right| = \frac{1}{|T(i\omega)|}, \quad \forall \omega \geq 0. \tag{3}$$

- **Condition** (3) **is one of the reasons why feedback systems work so well in practice.**
  - The models used to design control systems are often simplified, and the properties of a process may change during operation.
  - Condition (3) implies that the closed loop system will at least be stable for substantial variations in the process dynamics

The peak value of the complimentary sensitivity transfer function:

$$M_t = \max_{\omega} |T(i\omega)| := \left\| \frac{PC}{1 + PC} \right\|_{\infty}$$

- Condition (3) becomes $|\delta(i\omega)| < 1/M_t, \forall \omega \geq 0.$
- Reasonable values of M_t are from 1.2 to 2.

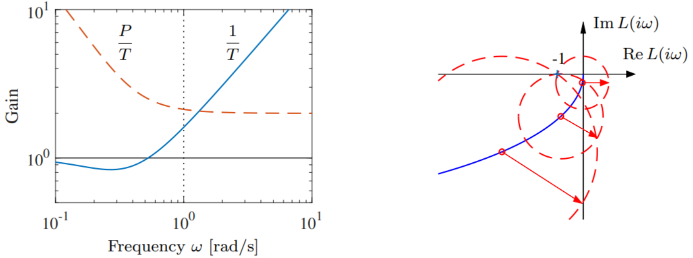
Example

Example (Example 13.7: Cruise Control)

The model of the car in the fourth gear at speed $20m/s$ is

$$P(s) = \frac{1.32}{s + 0.01}$$

- Consider a PI controller with gains $k_p = 0.5$ and $k_i = 0.1$.

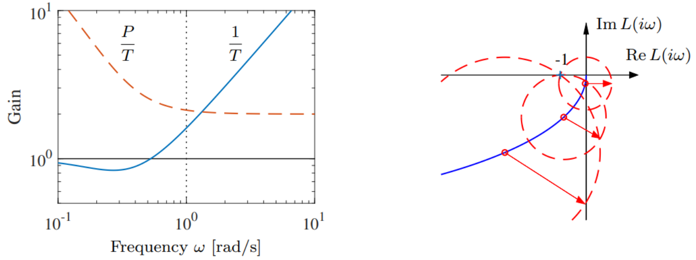


(a) Bounds on process uncertainty

(b) Nyquist plot representation of bounds

Figure: Robustness of a cruise controller

Example



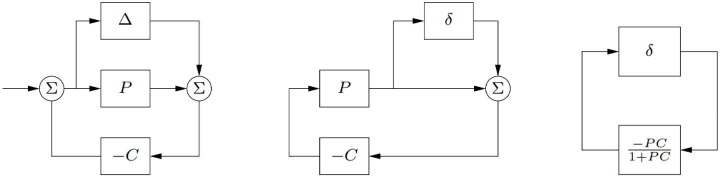
(a) Bounds on process uncertainty

(b) Nyquist plot representation of bounds

Some observations:

- Moderately small uncertainties are required only around the **gain crossover frequencies**,
- but large uncertainties can be permitted at higher and lower frequencies.
- A simple model that describes the process dynamics well around the crossover frequency is often sufficient for design

Other robustness conditions



(a) Additive uncertainty (b) Multiplicative uncertainty (c) Small gain

Figure: Illustration of robustness to process perturbations[1]

Table 13.1: Conditions for robust stability for different types of uncertainty.

Process	Uncertainty Type	Robust Stability
$P + \Delta$	Additive	$\|CS\Delta\|_\infty < 1$
$P(1+\delta)$	Multiplicative	$\|T\delta\|_\infty < 1$
$P/(1+\Delta_{\text{fb}} \cdot P)$	Feedback	$\|PS\Delta_{\text{fb}}\|_\infty < 1$

[1] The details of these sufficient conditions are not required in this class.

Outline

System design

The *initial design of a system* can have a significant impact on the ability to use feedback to provide **robustness** and **performance improvements**.

- It is particularly important to recognize fundamental limits in the performance of feedback systems early in the design process.
- Awareness of the limits and **co-design** of the process and the controller are good to avoid potential difficulties both for system and control designers.

Examples:

- We may expect that a system with **time delays** cannot admit fast control because control actions are delayed.
- It seems reasonable that **unstable systems will require fast controllers**, which will depend on the **bandwidth** of sensors and actuators.
- These limits are caused by properties of the system dynamics and can often be captured by conditions on the **poles and zeros** of the process

System design

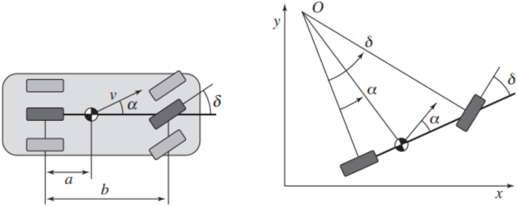
The freedom for the control designer depends very much on the situation

- **Extreme 1 (limited freedom)**: a process with given sensors and actuators and his or her task is to design a suitable controller
  - Even further, you may be given with an existing control loop, and your task is to retrofit the system.
- **Extreme 2 (significant freedom)**: You can choose sensors/actuators
  - **Co-design** the location and characteristics of sensors, actuators, and controller simultaneously.
  - However, you may have budget limits.

Performance limits due to dynamics and limits on actuation power/rate.

- **Dynamics limitations**: captured by time delays and poles and zeros in the right half-plane.
  - Time delays are easy to understand.
  - A less obvious case is that a process with a right half-plane pole/zero pair cannot be controlled robustly if the pole is close to the zero (details are not required in this class).
- **Restriction in actuation**: captured by actuation power and rates.

Example: Vehicle steering



(a) Overhead view

(b) Bicycle model

Figure: Vehicle steering dynamics. The left figure shows an overhead view of a vehicle with four wheels. The wheelbase is b.

- The center of mass at a distance a forward of the rear wheels.
- Approximation with a single front wheel and a single rear wheel — an abstraction called the **bicycle model**.
- The steering angle is δ and the velocity at the center of mass has the angle α relative the length axis of the vehicle.
- The position is given by (x, y) and the orientation (heading) by θ; For ODE modeling, see Example 3.11

Example: Vehicle steering

Example (Frequency modeling for vehicle steering)

The transfer function from steering angle δ to lateral position y is

$$P(s) = \frac{av_0 s + v_0^2}{bs^2}$$

- v_0 is the velocity of the vehicle and $a, b > 0$
- The transfer function has a zero

$$s = -\frac{v_0}{a}.$$

  - In normal (forward) driving this zero is in the left half-plane,
  - but it is in the right half-plane when driving in reverse ($v_0 < 0$).
- The unit step response is $y(t) = \frac{av_0}{b}t + \frac{v_0^2}{2b}t^2$
  - The lateral position thus begins to respond immediately to a steering command as an integrator.
  - If $v_0 < 0$ (reverse steering), the initial $y(t)$ is in the wrong direction!!
  - This behavior is representative for **non-minimum phase systems** (called an **inverse response**).

Example: Vehicle steering

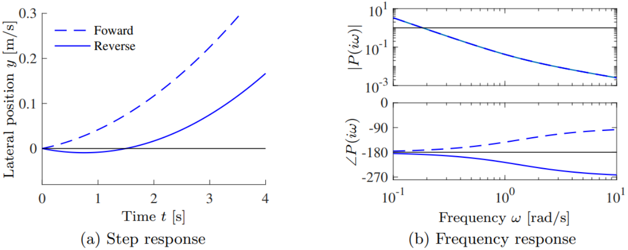
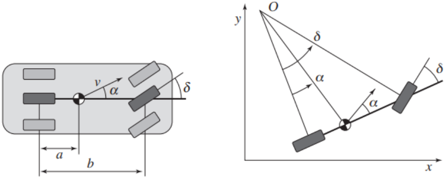


Figure: Vehicle steering for driving in reverse. (a) Step responses; (b) frequency responses

- The step response for forward and reverse driving is shown above.
- The parameters are a $a = 1.5m$, $b = 3m$, $v_0 = 2m/s$ for forward driving, and $v_0 = -2m/s$ for reverse driving.
- When driving in reverse, there is an initial motion of the center of mass in the opposite direction
- there is **A DELAY** before the car begins to move in the desired manner.

Example: Vehicle steering



(a) Overhead view (b) Bicycle model

The existence of the right half-plane zero can be removed

- if we choose to measure the location of the vehicle by the position of the center of the rear wheels instead of the center of mass

$$a = 0, \qquad P(s) = \frac{v_0^2}{bs}$$

- This is easily implemented by calibrating the position sensor for the vehicle
- This choice of "sensor" is subject to calibration errors ϵ and this can lead to a zero of the process transfer function at v_0/ϵ
- This is called a **"fast" zero** and its impact is relatively minor.

Poles and Zeros

- The poles of a system depend on the intrinsic dynamics of the system.
- They represent the modes of the system and they are given by the eigenvalues of the dynamics matrix A of the linearized model.
  - For example, we have the initial response to $\dot{x} = Ax$

$$x_i(t) = e^{\lambda_i t} x_i(0).$$

Sensors and actuators have no effect on the poles: **the only way to change poles is by feedback or by redesign of the process.**

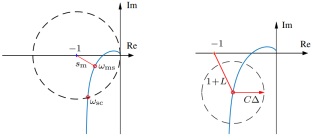
- However, the zeros of a system depend on how the sensors and actuators are connected to the process.
- Zeros can thus be changed by **moving or adding sensors and actuators**, which is often simpler than redesigning the process dynamics

Outline

Summary

- An explicit sufficient **robustness condition** based on Nyquist criterion

$$|C\Delta| < |1 + L|, \quad \text{or} \quad |\delta(i\omega)| < \left|\frac{1 + L(i\omega)}{L(i\omega)}\right| = \frac{1}{|T(i\omega)|}, \quad \forall \omega \geq 0.$$



(a) Nyquist plot (b) Additive uncertainty

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- **Performance limits** due to process dynamics and limits on actuation power/rate.
  - **Dynamics limitations**: captured by time delays and poles and zeros in the right half-plane.
    - Sensors and actuators have no effect on the poles: **the only way to change poles is by feedback or by redesign of the process.**
    - However, the zeros of a system depend on how the sensors and actuators are connected to the process.
  - **Restriction in actuation**: captured by actuation power and rates.