ECE 171A: Linear Control System Theory
Lecture 26: Fundamental Limits

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Reading materials: Chap 14.1, Chap 14.2

Final Exam

Final Exam — 8:00 am - 10:30 am, June 12

- **Scope**: Lectures 1 - 27, HW1 - HW8, DI 1-10; (Reading materials in the textbook)
- This final exam is closed book but you can bring **one sheet of notes**
  - page maximum size: Letter; can be double-sided;
  - should be hand-written (you can also write on your iPad and print it);
  - it is not acceptable to directly copy-paste lecture notes/HW/textbook.
- The exams must be done in a blue book. Bring a blue book with you.
- No MATLAB is required. No graphing calculators are permitted. You need a basic arithmetic calculator for simple calculations.
- **No collaboration and discussions are allowed**. It is dishonest to cheat on exams. Instances of academic dishonesty will be referred to the Office of Student Conduct for adjudication. *You don't want to take a risk for such a small thing*.

Student Evaluations of Teaching (SET)

- You should have got the following link from UCSD Online Evaluations to evaluate ECE 171A

  `https://academicaffairs.ucsd.edu/Modules/Evals/?e11360527`
- **Deadline**: Saturday, June 08 at 8:00 am
- Your responses are completely anonymous.
- Please give some **thoughtful** and **constructive** feedback.
- It's your opportunity to let your voices be heard (by our team, the department, and the university).
- If you like the course, please say it explicitly and we'd love to hear it
- If you think some aspects can be further improved, we are more than happy to know them (we have implemented some suggestions from previous surveys)

Many thanks for your efforts and your time in this course!

Outline

Outline

System design

The *initial design of a system* can have a significant impact on the ability to use feedback to provide **robustness** and **performance improvements**.

- It is particularly important to recognize fundamental limits in the performance of feedback systems early in the design process.
- Awareness of the limits and **co-design** of the process and the controller are good to avoid potential difficulties both for system and control designers.

Examples:

- We may expect that a system with **time delays** cannot admit fast control because control actions are delayed.
- It seems reasonable that **unstable systems will require fast controllers**, which will depend on the **bandwidth** of sensors and actuators.
- These limits are caused by properties of the system dynamics and can often be captured by conditions on the **poles and zeros** of the process

System design

The freedom for the control designer depends very much on the situation

- **Extreme 1 (limited freedom)**: a process with given sensors and actuators and his or her task is to design a suitable controller
  - Even further, you may be given with an existing control loop, and your task is to retrofit the system.
- **Extreme 2 (significant freedom)**: You can choose sensors/actuators
  - **Co-design** the location and characteristics of sensors, actuators, and controller simultaneously.
  - However, you may have budget limits.

Performance limits due to dynamics and limits on actuation power/rate.

- **Dynamics limitations**: captured by time delays and poles and zeros in the right half-plane.
  - Time delays are easy to understand.
  - A less obvious case is that a process with a right half-plane pole/zero pair cannot be controlled robustly if the pole is close to the zero (details are not required in this class).
- **Restriction in actuation**: captured by actuation power and rates.

Example: Vehicle steering

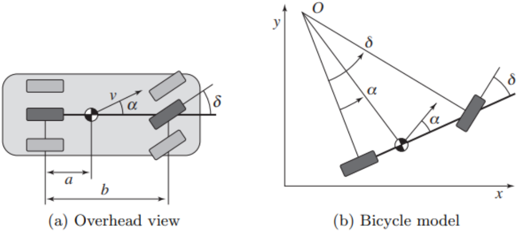


Figure: Vehicle steering dynamics. The left figure shows an overhead view of a vehicle with four wheels. The wheelbase is b.

- The center of mass is at a distance a from the rear wheels.
- Approximation with a single front wheel and a single rear wheel — an abstraction called the **bicycle model**.
- The steering angle is δ and the velocity at the center of mass has the angle α relative to the length axis of the vehicle.
- The position is given by (x, y) and the orientation (heading) by θ; For ODE modeling, see Example 3.11

Example: Vehicle steering

Example (Frequency modeling for vehicle steering)

The transfer function from steering angle δ to lateral position y is

$$P(s) = \frac{av_0 s + v_0^2}{bs^2}$$

- v_0 is the velocity of the vehicle and $a, b > 0$
- The transfer function has a zero

$$s = -\frac{v_0}{a}.$$

  - In normal (**forward**) driving this zero is in the left half-plane,
  - but it is in the right half-plane when driving in **reverse** ($v_0 < 0$).
- The unit step response is $y(t) = \frac{av_0}{b}t + \frac{v_0^2}{2b}t^2$
  - The lateral position thus begins to respond immediately to a steering command as an integrator.
  - If $v_0 < 0$ (reverse steering), the initial $y(t)$ is in the wrong direction!!
  - This behavior is representative for **non-minimum phase systems** (Chap 10.4; called an **inverse response**).

Example: Vehicle steering

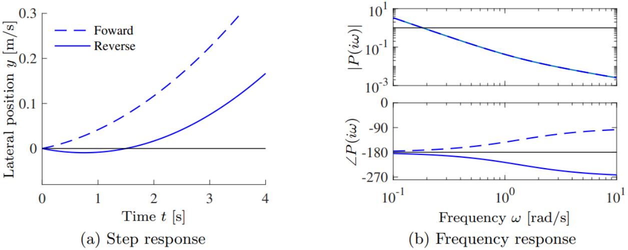
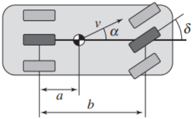


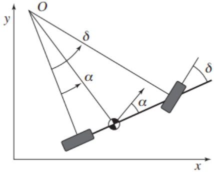
Figure: Vehicle steering for driving in reverse. (a) Step responses; (b) frequency responses

- The step response for forward and reverse driving is shown above.
- The parameters are a $a = 1.5m$, $b = 3m$, $v_0 = 2m/s$ for forward driving, and $v_0 = -2m/s$ for reverse driving.
- When driving in **reverse**, there is an initial motion of the center of mass in the **opposite** direction
- There is **A DELAY** before the car begins to move in the desired manner.

Example: Vehicle steering



(a) Overhead view



(b) Bicycle model

The existence of the right half-plane zero can be removed

- if we choose to measure the location of the vehicle by the position of the center of the rear wheels instead of the center of mass

$$a = 0, \qquad P(s) = \frac{v_0^2}{bs}$$

- This is easily implemented by calibrating the position sensor for the vehicle
- This choice of "sensor" is subject to calibration errors ϵ and this can lead to a zero of the process transfer function at v_0/ϵ
- This is called a **"fast" zero** and its impact is relatively minor.

Poles and Zeros

- The poles of a system depend on the intrinsic dynamics of the system.
- They represent the modes of the system and they are given by the eigenvalues of the dynamics matrix A of the linearized model.
  - For example, we have the initial response to $\dot{x} = Ax$

$$x_i(t) = e^{\lambda_i t} x_i(0).$$

Sensors and actuators have no effect on the poles: **the only way to change poles is by feedback or by redesign of the process.**

- However, the zeros of a system depend on how the sensors and actuators are connected to the process.
- In general, zeros can be changed by **moving or adding sensors and actuators**, which is often simpler than redesigning the process dynamics

Outline

Bode's Integral formula

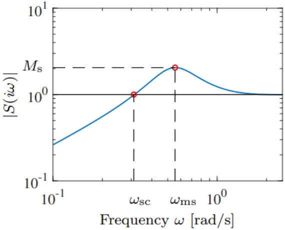
One of the most important limits in feedback control was obtained by Bode,

- It is **IMPOSSIBLE** to uniformly improve the performance of certain closed-loop performance characteristics.
- The sensitivity function

$$S = \frac{1}{1 + P(s)C(s)}$$

shows how feedback $C(s)$ influences the response of the output to disturbances w

- Disturbances with frequencies such that $|S(i\omega)| < 1$ are **attenuated**; such that $|S(i\omega)| > 1$ are **amplified** by feedback

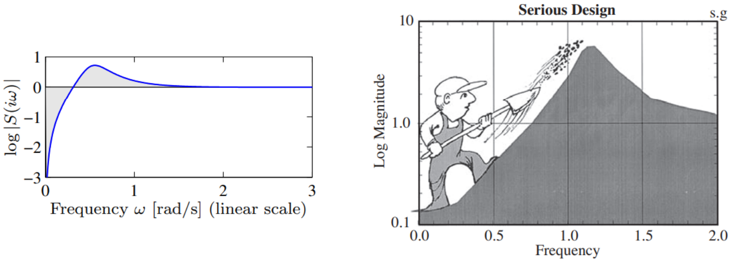


(a) Gain curves

The sensitivity function cannot be made small over a wide frequency range.

- There is an invariant (conserved quantity) called *Bode's integral formula*.
- It implies that reducing the sensitivity at one frequency increases it at another
- The situation is worse if the process has **right half-plane poles**.

Waterbed Effect



(a) Bode integral formula (b) Control design process

Figure: Interpretation of the **waterbed effect**

- The function $\log|S(i\omega)|$ is plotted versus ω using a linear scale in (a).
- According to Bode's integral formula, the area of $\log|S(i\omega)|$ above zero must be equal to the area below zero.
- Gunter Stein's interpretation[1] of design as a trade-off of sensitivities at different frequencies is shown in (b)

[1]This talk by Prof. Stein is highly recommended: `https://youtu.be/9Lhu31X94V4`

Bode's Integral formula

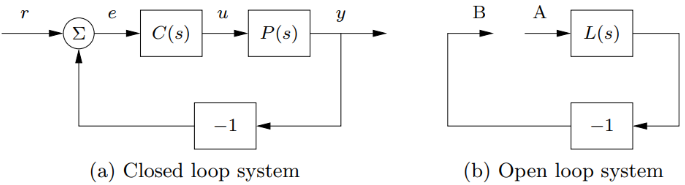


Figure: **The loop transfer function $L(s) = P(s)C(s)$.**

Theorem

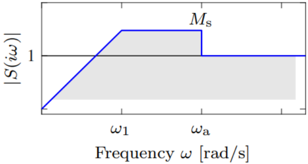
Let $S(s)$ be the sensitivity function of an internally stable closed loop system with loop transfer function $L(s)$. Assume that the loop transfer function $L(s)$ is such that $sL(s)$ goes to zero as $s \to \infty$. Then the sensitivity function has the property

$$\int_0^\infty \log |S(i\omega)| d\omega = \int_0^\infty \log \frac{1}{|1 + L(i\omega)|} d\omega = \pi \sum p_k$$

where p_k are the right half-plane poles of $L(s)$.

Example: X-29 aircraft

(a) X-29 aircraft



(b) Sensitivity analysis

Figure: (a) X-29 flight control system; (b) The desired sensitivity for the closed-loop system

- This analysis was originally carried out by Gunter Stein in his inaugural IEEE Bode lecture "Respect the Unstable"
- Longitudinal dynamics of X-29 are similar to inverted pendulum dynamics
  - It has a right half-plane pole at $p \approx 6$ rad/s and a right half-plane zero at $z = 26$ rad/s.
  - The actuators that stabilize the pitch have a bandwidth of $\omega_a = 40$ rad/s.
  - The desired bandwidth of the pitch control loop is $\omega_1 = 3$ rad/s.

Example: X-29 aircraft

To evaluate the achievable performance, we search for a control law such that the sensitivity function is small up to the desired bandwidth ω_1 and not greater than M_s beyond that frequency

- **Bode's integral formula** implies that $M_s > 1$ at high frequencies to balance the small sensitivity at low frequency
- We assume that the sensitivity function is given by

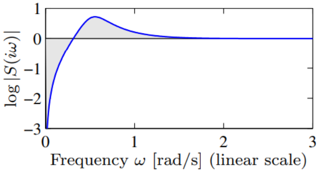
$$|S(i\omega)| = \begin{cases} \frac{\omega}{\omega_1} M_s & \text{if } \omega < \omega_1 \\ M_s & \text{if } \omega_1 \leq \omega < \omega_a \\ 1 & \text{if } \omega_a \leq \omega < \infty \end{cases}$$

- From Bode's integral formula, we have

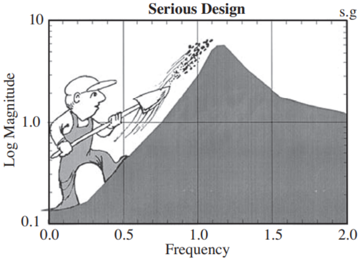
$$\int_0^\infty \log |S(i\omega)| = \pi p,$$

  where p is the open-loop unstable pole.
- After some calculation, we get $M_s = e^{(\pi p + \omega_1)/\omega_2}$ — **performance limits**.

The first Bode Talk: Respect the Unstable



(a) Bode integral formula



(b) Control design process

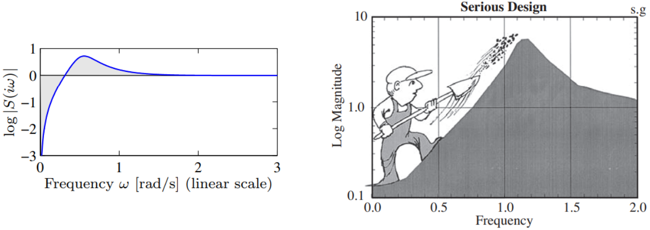
Figure: Interpretation of the **waterbed effect**

The first Bode Talk — **Respect the Unstable**:
`https://youtu.be/9Lhu31X94V4`

Outline

Summary

- **Performance limits** due to process dynamics and limits on actuation power/rate.
  - **Dynamics limitations**: captured by time delays and poles and zeros in the right half-plane.
    - Sensors and actuators have no effect on the poles: **the only way to change poles is by feedback or by redesign of the process.**
    - However, the zeros of a system depend on how the sensors and actuators are connected to the process.
  - **Restriction in actuation**: captured by actuation power and rates.
- Bode's integral formula



(a) Bode integral formula

(b) Control design process

Figure: Interpretation of the **waterbed effect**