ECE 171A: Linear Control System Theory
Lecture 27: Review

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Final Exam

Final Exam — 8:00 am - 10:30 am, June 12

- **Scope**: Lectures 1 - 27, HW1 - HW8, DI 1-10; (Reading materials in the textbook)
- This final exam is closed book but you can bring **one sheet of notes**
  - page maximum size: Letter; can be double-sided;
  - should be hand-written (you can also write on your iPad and print it);
  - it is not acceptable to directly copy-paste lecture notes/HW/textbook.
- The exams must be done in a blue book. Bring a blue book with you.
- No MATLAB is required. No graphing calculators are permitted. You need a basic arithmetic calculator for simple calculations.
- **No collaboration and discussions are allowed**. It is dishonest to cheat on exams. Instances of academic dishonesty will be referred to the Office of Student Conduct for adjudication. *You don't want to take a risk for such a small thing*.

Extra Office Hours

- Next Tuesday, June 11 - 4:30 pm to 8:30 pm (FAH 3002)

Student Evaluations of Teaching (SET)

- You should have got the following link from UCSD Online Evaluations to evaluate ECE 171A

  `https://academicaffairs.ucsd.edu/Modules/Evals/?e11360527`
- **Deadline**: Saturday, June 08 at 8:00 am
- Your responses are completely anonymous.
- Please give some **thoughtful** and **constructive** feedback.
- It's your opportunity to let your voices be heard.
- If you like the course, please say it explicitly and we'd love to hear it
- If you think some aspects can be further improved, we are more than happy to know them (we have implemented some suggestions from previous surveys)

Course Participation: Survey on Canvas

- Bonus 1: 1% credit to your class participation (maximum 5%).
- Bonus 2: If the participation rate reaches 80%, we will add a bonus question in the final exam, similar to Midterms.

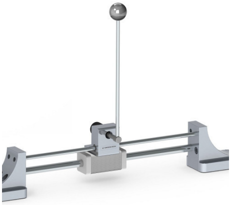
Outline

Outline

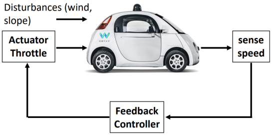
Lecture 1 - Overview of control systems

A **control system** is an interconnection of two or more dynamical systems that provides a **desired response**.

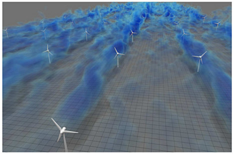
- Control is to modify the inputs to the plant to produce a **desired output**.



(a) Inverted Pendulum



(b) Cruise control



(c) Wind farm

- Feedforward control vs. feedback control
- Two live experiments
- **Feedback control** = **Sensing** + **Computation** + **Actuation**

Lecture 2 - ODEs and the first control example

Review on ODEs

- An n**th-order linear ordinary differential equation (ODE)** is:

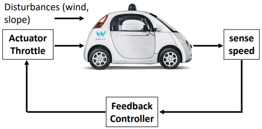
$$\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_1\frac{d}{dt}y(t) + a_0 y(t) = u(t)$$

- First-order matrix ODE

$$\dot{x}(t) = Ax(t) + Bu(t)$$

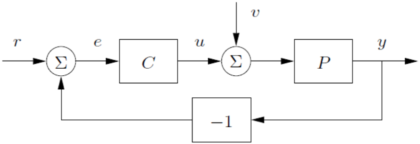
Cruise control

P control $F_{\text{engine}} = K_{\text{p}} e(t)$
I control $F_{\text{engine}} = K_{\text{i}} \int_0^t e(t)dt$
D control $F_{\text{engine}} = K_{\text{d}} \frac{d}{dt} e(t)$



Feedback control $=$ Sensing $+$ Computation $+$ Actuation

Lecture 3 - Feedback principles



- We have considered static plant dynamics with analytical solutions

$$y = \mathrm{sat}(x) = \begin{cases} -1 & \text{if } x \leq -1 \\ x & \text{if } |x| < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

- and a simple dynamical model with numerical simulations
- to illustrate several fundamental properties of feedback

  - Disturbance attenuation
  - Reference signal tracking
  - Robustness to uncertainty
  - Shaping of dynamical behavior

Lecture 4/5 - System modeling

A model is a *mathematical representation* of a physical, biological, or information system.

- Models allow us to reason about a system and make predictions about how a system will behave.
- The model we choose depends on the questions we wish to answer, and so there may be multiple models for a single dynamical system.
- It is important to keep in mind that all models are an approximation of the underlying system.

The choice of state is not unique.

- There may be many choices of variables that can act as the state.
  - A trivial example: One can choose different units (scaling factors)
  - A less trivial example: One can take sums and differences of some variables.

Lecture 4/5 - System modeling

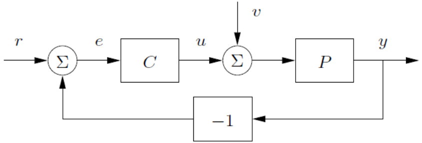
- $x \in \mathbb{R}^n$: state; $y \in \mathbb{R}^p$: output; $u \in \mathbb{R}^m$: input
- Continuous-time (e.g., speed control, inverted pendulum, spring-mass)

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du. \end{aligned}$$

- Discrete-time (e.g., consensus protocol, predator-prey dynamics)

$$\begin{aligned} x[k+1] &= f(x[k], u[k]), \\ y[k] &= h(x[k], u[k]). \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} x[k+1] &= Ax[k] + Bu[k], \\ y[k] &= Cx[k] + Du[k]. \end{aligned}$$

- **Block diagrams**: Emphasize the information flow and hide details of the system.

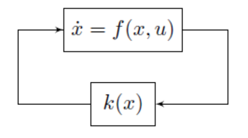


Lecture 6 - System solutions and Phase portraits

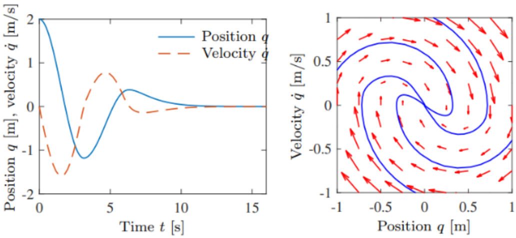
Closed-loop system: with $u = k(x)$

$$\dot{x}(t) = f(x, k(x)) := F(x).$$

Analytical or *Computational* solutions



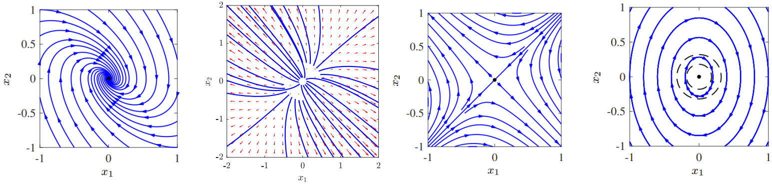
- **Solving differential equations**
- **Qualitative analysis**: phase portraits and time plot



(a) Time plot (b) Phase portrait

Lecture 7 - Equilibrium and stability

- An **equilibrium** point of a dynamical system represents a *stationary* condition for the dynamics.
- Stable, asymptotically stable, unstable; terminology like **sink**, **source**, **saddle**, **center**



(a) Sink (b) Source (c) Saddle (d) Center

- Stability of linear systems $\dot{x}(t) = Ax(t)$
  - Eigenvalue test
  - **Routh–Hurwitz** Criterion

Lecture 8: Jacobian Linearization

Consider a nonlinear system $\dot{x} = F(x)$, with $x_{\mathrm{e}} = 0$ as an equilibrium point. Let

$$A = \left.\frac{\partial F}{\partial x}\right|_{x_{\mathrm{e}}=0}$$

- $x_{\mathrm{e}} = 0$ is locally asymptotically stable if A is asymptotically stable or all eigenvalues of A have negative real parts.
- $x_{\mathrm{e}} = 0$ is unstable if one or more of the eigenvalues of A has positive real part.

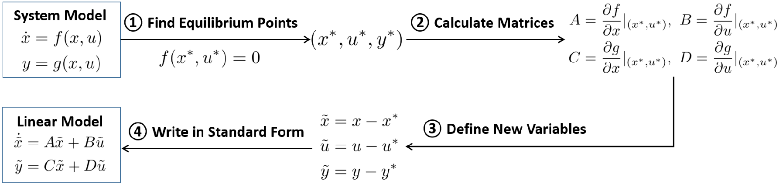


Figure: Model Linearization Procedure (Taken from Prof Na Li's ES 155)

L10: Input-output response (I)

- The output $y(t)$ of an LIT system has very nice **linear properties**:
  - Zero initial state $x(0) = 0$: *the output $y(t)$ is linear in input $u(t)$*;
  - Zero input $u(t) = 0$: *the output $y(t)$ is linear in initial states $x(0)$*.
- Initial response – **matrix exponential**:
  - The solution to $\dot{x} = Ax, x(0) \in \mathbb{R}^n$ is given by $x(t) = e^{At}x(0)$.
- Three very important test signals:
  - **Step input** $u(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t > 0 \end{cases}$
  - **Impulse input**

$$u(t) = p_\epsilon(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1/\epsilon & \text{if } 0 \leq t < \epsilon \\ 0 & \text{if } t \geq \epsilon \end{cases} \qquad \delta(t) = \lim_{\epsilon \to 0} p_\epsilon(t)$$

  - **Frequency input**

$$u(t) = \sin(\omega t + \phi).$$

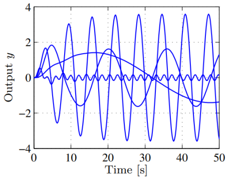
Outline

L11 - Input/output responses (II)

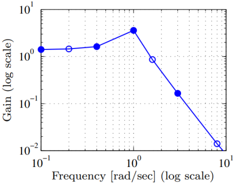
- **Impulse response**

$$h(t) = \int_0^t Ce^{A(t-\tau)}B\delta(\tau)d\tau + D\delta(t) = Ce^{At}B + D\delta(t).$$

- **Frequency responses**



(a) Time domain simulations



(b) Frequency response

- **The convolution equation**

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t).$$

another version is $y(t) = \underbrace{Ce^{At}x(0)}_{\text{Initial response}} + \underbrace{\int_0^t h(t-\tau)u(\tau)d\tau}_{\text{forced response}}$

L12: Transfer function (I)

- Transient response and steady-state response

$$y(t) = \underbrace{Ce^{At}\left(x(0) - (sI - A)^{-1}B\right)}_{\text{transient}} + \underbrace{\left(C(sI - A)^{-1}B + D\right)e^{st}}_{\text{steady-state}}$$

- Transfer function

$$G(s) = C(sI - A)^{-1}B + D.$$

  - Express the steady-state solution of a **stable** linear system forced by a sinusoidal input

$$u(t) = \sin(\omega t) \to y_{\text{ss}} = |G(i\omega)| \sin(\omega t + \angle G(i\omega))$$

- **Frequency domain modeling**: Modeling a system through its response to sinusoidal and exponential signals.
  - We represent the dynamics of the system in terms of the generalized frequency s rather than the time domain variable t.
  - The **transfer function** provides a complete representation of a linear system in the frequency domain.

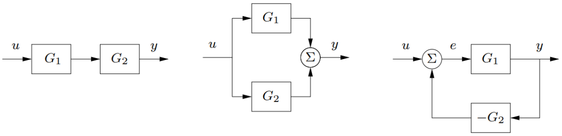
L13: Transfer function (II)

- Transfer function for linear ODEs

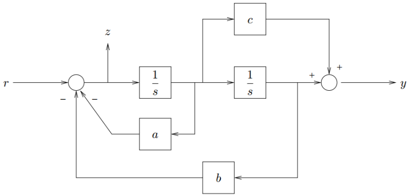
$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1} u}{dt^{m-1}} + \ldots + b_0 u,$$

$$G(s) = \frac{b_m s^m + b_{m-1} s^{n-1} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}.$$

- Block diagram with transfer functions



(a) $G_{yu} = G_2 G_1$ (b) $G_{yu} = G_1 + G_2$ (c) $G_{yu} = \dfrac{G_1}{1 + G_1 G_2}$



L14: Zeros, Poles and Bode plot

- The **features** of a transfer function are often associated with **important system properties**.
  - zero frequency gain
  - the locations of the poles and zeros: **Poles** — modes of a system; **Zeros** – Block transmission of certain signals

Poles (eigenvalues) of the matrix A = Poles of the transfer function $G(s)$

- The frequency response $G(i\omega)$ can be represented by two curves — **Bode plot**
  - **Gain curve**: gives $|G(i\omega)|$ as a function of frequency ω — log/log scale (traditionally in dB — $20 \log |G(i\omega)|$; we often consider $\log |G(i\omega)|$)
  - **Phase curve**: gives $\angle G(i\omega)$ as a function of frequency ω — log/linear scale in degrees

L15: Bode plot and Routh-Hurwitz stability

- The **Bode plot** gives a quick overview of a **stable** linear system.

$$u(t) = \sin(\omega t) \qquad \rightarrow \qquad y_{\mathrm{ss}} = |G(i\omega)| \sin(\omega t + \angle G(i\omega))$$

Theorem

Consider a Routh table from the polynomial $a(s)$ in

$$G(s) = \frac{b(s)}{a(s)}.$$

- *The number of sign changes in the first column of the Routh table is equal to the number of roots of $a(s)$ in the closed right half-plane.*

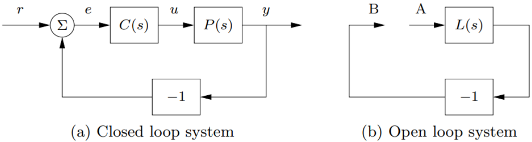
Corollary (BIBO Stability of LTI Systems)

The system $G(s)$ is **BIBO stable** *if and only if there are no sign changes in the first column of its Routh table.*

L16: Loop transfer functions and Nyquist plot

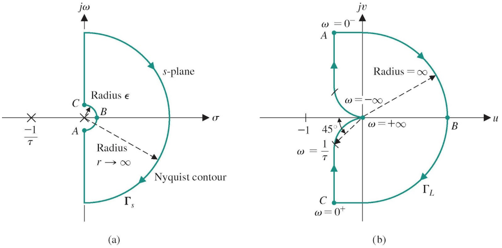
- Nyquist's idea was to first investigate conditions under which oscillations can occur in a feedback loop – The **Loop transfer function**:

$$L(s) = P(s)C(s).$$



(a) Closed loop system (b) Open loop system

- **Nyquist plot** and Simplified **Nyquist criterion**

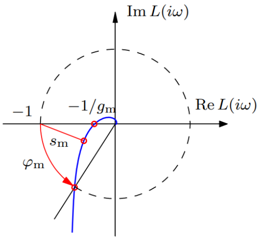


L17: Nyquist plot and Nyquist Criterion

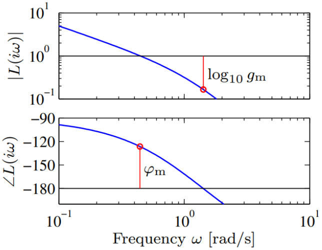
Theorem (Nyquist's Stability Criterion)

Consider a unity feedback control system with open-loop transfer function $L(s)$. Let Γ be a Nyquist contour. The system is stable if and only if the number of counterclockwise encirclements of $-1 + i0$ by the Nyquist plot $L(\Gamma)$ is equal to the number of poles of $L(s)$ inside Γ.

Classical robustness measures: stability margin, phase margin, gain margin



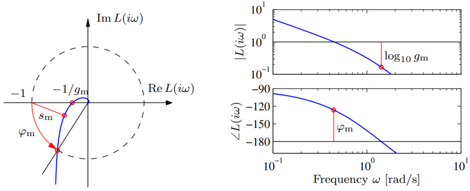
(a) Nyquist plot



(b) Bode plot

L18/L19: Stability margins and Root locus

- **Stability margins** express how well the Nyquist curve of the loop transfer avoids the critical point -1.
- The shortest distance s_m of the Nyquist curve to the critical point is a natural criterion — **stability margin**; Another two criteria are **gain margin** and **phase margin**.



(a) Nyquist plot (b) Bode plot

- **Root locus**: a graph of the closed-loop roots as k is varied from 0 to ∞.

  - The plot of root locus will have n branches.
  - Each branch starts at a different open-loop pole.
  - m of the branches end at different open-loop zeros.
  - The remaining $n-m$ branches go to infinity.

L20: PID control

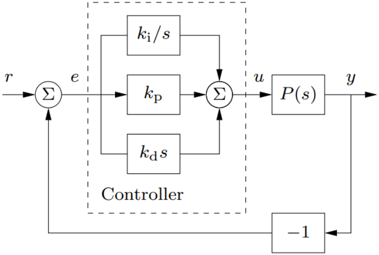


Figure: PID using error feedback

PID control

- the proportional term (P) — the **present** error;
- the integral term (I) — the **past** errors;
- the derivative term (D) — anticipated **future** errors.

- **Magic of integral action**

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau) d\tau.$$

$$\Rightarrow \quad u_0 = k_p e_0 + k_i \lim_{t \to \infty} \int_0^t e(\tau) d\tau.$$

- PID controller design for first-order and second-order systems

Outline

L22: Performance specifications

- **Sensitivity functions**: for most control designs we focus on the following subset — the **Gang of Six**

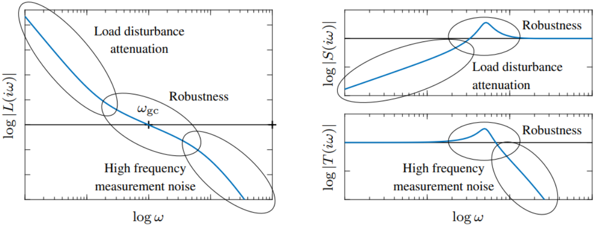
$$G_{\text{yr}} = \frac{PCF}{1+PC}, \quad -G_{\text{uv}} = \frac{PC}{1+PC}, \quad G_{\text{yv}} = \frac{P}{1+PC}$$
$$G_{\text{ur}} = \frac{CF}{1+PC}, \quad -G_{\text{uw}} = \frac{C}{1+PC}, \quad G_{\text{yw}} = \frac{1}{1+PC}$$

- Specifications capture **robustness** to process variations and **performance** w.r.t.
  - the ability to **track** reference signals and **attenuate** load disturbances **without injecting** too much measurement noise.

- Expressed in terms of transfer functions (e.g., the Gang of Six) and the loop transfer function, using features of their **time and frequency responses**.

L23: Loop Shaping

- The **loop transfer function** should have roughly the shape below



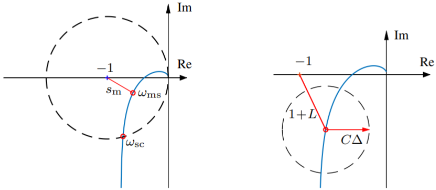
(a) Gain plot of loop transfer function (b) Gain plot of sensitivity functions

- General purpose of **Lag compensation**
  - increases the gain at low frequencies
  - improve tracking performance at low frequencies
  - improve disturbance attenuation at low frequencies
- General purpose of **Lead compensation**
  - Add phase lead in the frequency range between the pole and zero pair
  - By appropriately choosing the location of this phase lead, we can provide **additional phase margin** at the gain crossover frequency.

L24: Robustness and uncertainty

- **Robustness to uncertainty** is one of the most useful properties of feedback — design feedback systems based on *strongly simplified models*.
  - **Parametric uncertainty** in which the parameters describing the system are not precisely known
  - **Unmodeled dynamics**, in which some dynamics are neglected during the modeling.
- An explicit sufficient **robustness condition** based on Nyquist criterion

$$|C\Delta| < |1+L|, \quad \text{or} \quad |\delta(i\omega)| < \left|\frac{1+L(i\omega)}{L(i\omega)}\right| = \frac{1}{|T(i\omega)|}, \quad \forall \omega \geq 0.$$



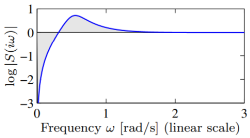
(a) Nyquist plot (b) Additive uncertainty

Figure 13.7: Illustrations of robust stability in Nyquist plots. The plot (a) shows the stability margin $s_m = 1/M_s$. The plot (b) shows the Nyquist curve and the circle shows uncertainty due to stable additive process variations Δ.

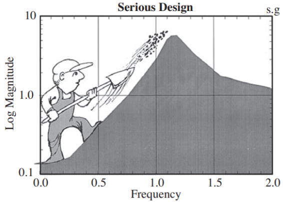
L25/26: Fundamental Limits

- **Performance limits** due to process dynamics and limits on actuation power/rate.
  - **Dynamics limitations**: captured by time delays and poles and zeros in the right half-plane.
    - Sensors and actuators have no effect on the poles: **the only way to change poles is by feedback or by redesign of the process.**
    - However, the zeros of a system depend on how the sensors and actuators are connected to the process.
  - **Restriction in actuation**: captured by actuation power and rates.
- **Waterbed Effect** — Bode's integral formula

$$\int_0^\infty \log|S(i\omega)|d\omega = \int_0^\infty \log\frac{1}{|1+L(i\omega)|}d\omega = \pi\sum p_k$$



(a) Bode integral formula



(b) Control design process

Outline

Background survey from Week 1

- Are there any specific applications of feedback and control concepts that you are interested in?
  - Autonomous vehicles, robotics, and humanoid robots.
  - automatic center of mass adjustment robot
  - Electric vehicles
  - Machine and reinforcement learning
  - ......
- Suggestions we have got so far
  - Some coding example during lecture or DI.
  - Responsive on Piazza
  - Putting student wellbeing first is very much appreciated.
  - examples in class that can help with better understanding concepts/coursework
  - Engaging and available outside the class
  - In-depth coverage of bode plots and related topics
  - ......

We hope this course has met your expectations!

Final words

- Control is an old yet fascinating area; control is everywhere;
- I hope you all have learned something from this class
- I hope you all can do very well in your final(s).
- Work hard and play hard (enjoy the summer vacation)!
- Feel free to reach out even after this course finishes!
- Good luck with your studies at UC San Diego and beyond!
- I look forward to your great news.

Thank you very much for your efforts and for staying to the end of this course!