ECE 171A: Linear Control System Theory
Lecture 3: Feedback Principles

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April 05, 2024

Reading materials: Ch 2.1, Ch 2.3, 2.4, 2.5

Power of feedback

Fundamental properties of feedback:

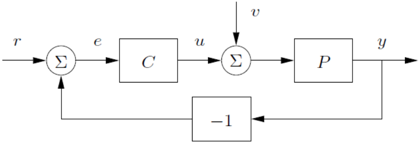
- Disturbance attenuation
- Reference signal tracking
- Robustness to uncertainty
- Shaping of dynamical behavior

We provide two simple examples to illustrate the properties above:

- A simple static model
- A simple dynamical model - Cruise control

Outline

A nonlinear static model



- Here, we consider static plant dynamics (which has no dynamical behavior; no ODE is needed)

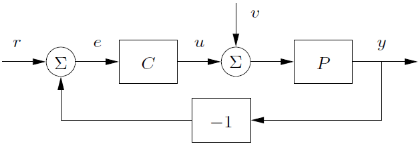
$$y = \text{sat}(x) = \begin{cases} -1 & \text{if } x \leq -1 \\ x & \text{if } |x| < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

- The controller C is a constant gain, i.e., $u = ke$ with $k > 0$.
- **Linear range**: the plant process is linear if $|x| < 1$, where we have $y = x$ and the process gain is 1.
- **Open-loop system**: a combination of the controller and the process with no feedback (assuming $v = 0$) leads to

$$y = \text{sat}(kr).$$

Its linear range becomes $|r| < 1/k$.

Response to Reference Signals



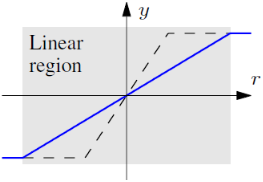
With the feedback loop, we have the closed-loop system (assuming $v = 0$)

$$\begin{cases} y = \mathrm{sat}(u), \\ u = k(r - y). \end{cases}$$

$$\Rightarrow \quad y = \mathrm{sat}(k(r - y))$$

- The overall input/output relationship becomes

$$y = \mathrm{sat}\left(\frac{k}{k+1} r\right) = \begin{cases} -1 & \text{if } r \leq -\frac{k+1}{k} \\ \frac{k}{k+1} r & \text{if } |r| < \frac{k+1}{k} \\ 1 & \text{if } r \geq \frac{k+1}{k} \end{cases}$$

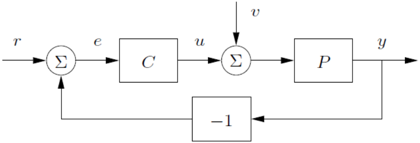


- **Linear range** of the closed-loop system is

$$|r| < \frac{k+1}{k}$$

Observation 1: Negative feedback **widens** the linear range of the system by a factor of $k + 1$ compared to the open loop system (that is $1/k$).

Robustness to Parameter Uncertainty



The **sensitivity** of a system describes how changes in the system parameters affect the performance of the system.

Case 1: Open-loop system: in the linear range, we have $y = kr$

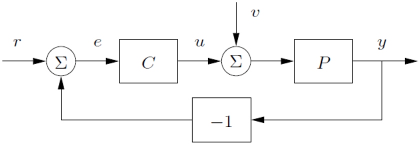
- It follows that
$$\frac{dy}{dk} = r = \frac{y}{k} \quad \Rightarrow \quad \frac{dy}{y} = \frac{dk}{k}$$
- **Sensitivity**: 10% change in k will lead to a 10% change in the output.

Case 2: Closed-loop system: in the linear range, we have $y = \frac{k}{k+1} r$

- $$\frac{dy}{dk} = \frac{r}{k+1} - \frac{kr}{(k+1)^2} = \frac{r}{(k+1)^2} = \frac{y}{k(k+1)} \quad \Rightarrow \quad \frac{dy}{y} = \frac{1}{k+1}\frac{dk}{k}$$
- **Sensitivity**: If $k = 100$, then 10% change in k will lead to less than a 0.1% change in y.

Observation 2: Negative feedback **reduces the sensitivity** to gain variations by a factor of $k+1$; the closed-loop system is much less sensitive.

Load Disturbance Attenuation



Another use of feedback is to **reduce** the effects of external disturbances, represented by the signal v in our case.

Case 1: Open-loop system: we have $y = \text{sat}(kr + v)$

- **Effect of v:** in the linear range, disturbances are passed through with no attenuation!

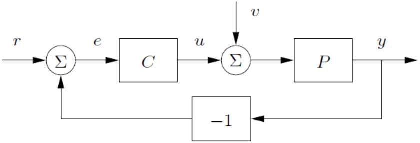
Case 2: Closed-loop system:

- For simplicity, we set the reference signal $r = 0$;
- Then, we have

$$y = \text{sat}(v - ky) \quad \Rightarrow \quad y = \text{sat}\left(\frac{v}{k+1}\right)$$

Observation 3: In the linear range, negative feedback *reduces the effect of load disturbances* by a factor of $k + 1$.

Summary



Static plant dynamics

$$y = \mathrm{sat}(x) = \begin{cases} -1 & \text{if } x \leq -1 \\ x & \text{if } |x| < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

Constant gain, i.e., $u = ke$ with $k > 0$.

Negative feedback

- 1) *increases* the range of linearity of the system,
- 2) *decreases* the sensitivity of the system to parameter variation,
- 3) *attenuates* load disturbances.

The **trade-off** is that the closed-loop gain is decreased

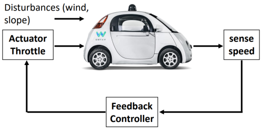
$$y = \mathrm{sat}\left(\frac{k}{k+1}r\right) = \begin{cases} -1 & \text{if } r \leq -\frac{k+1}{k} \\ \dfrac{k}{k+1}r & \text{if } |r| < \frac{k+1}{k} \\ 1 & \text{if } r \geq \frac{k+1}{k} \end{cases}$$

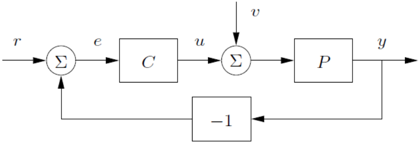
Outline

Cruise control

Parameters, input/output variables (simplified)

- Desired speed: v_{des}
- System variable (output): speed v
- System parameter: mass m (which may change)
- Disturbance: road slop $F_{\text{hill}} = -mg\sin(\theta)$, air drag $F = -\delta \times v$
- Actuator (input): Engine/Braking Force F_{engine}





System model

$$m\dot{v} = F_{\text{engine}} - \delta \times v - mg\sin(\theta)$$

PI control

$$F_{\text{engine}} = K_{\text{p}}e(t) + K_{\text{i}}\int_0^t e(t)dt, \qquad \text{where } e(t) = v_{\text{des}}(t) - v(t)$$

Reducing the effects of disturbances

Reducing the effects of disturbances is a primary use of feedback.

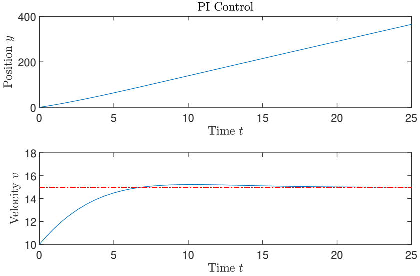
- It was used by James Watt to make steam engines run at constant speed in spite of varying load (Industrial revolution)
- It was used by electrical engineers to make generators driven by water turbines deliver electricity with constant frequency and voltage.
- Feedback is commonly used to alleviate effects of disturbances in the process industry, for machine tools, and for engine and cruise control in cars.
- The human body exploits feedback to keep body temperature, blood pressure, and other important variables constant.

Regulation problem: Keeping variables close to a desired, constant reference value in spite of disturbances.

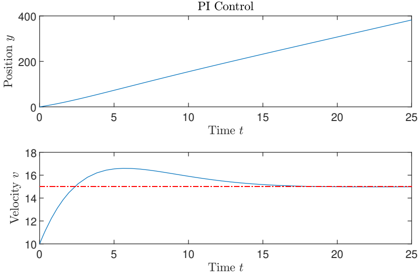
No steady-state error

Condition: $v_0 = 10m/s$, $m = 500\text{kg}, \delta = 0.5, \theta = 0$
PI controller: $K_\text{p} = 250$, $K_\text{i} = 50$

Case 1: Uphill $\theta = 5^\circ$



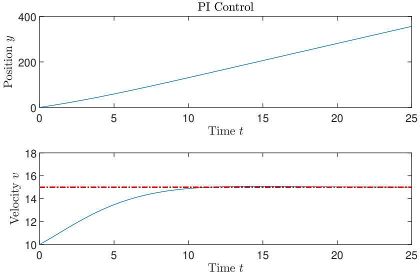
Case 2: Downhill $\theta = -5^\circ$



No steady-state error

Condition: $v_0 = 10m/s$, $m = 500\text{kg}, \delta = 0.5, \theta = 0$
PI controller: $K_\text{p} = 250,\ K_\text{i} = 50$

Case 3: Larger uphill $\theta = 10^\circ$



Disturbance attenuation: The same PI controller gives *no steady-state error*

$$e(t) = v_\text{des} - v \to 0,$$

given a constant disturbance (the value can be unknown to the controller).

Outline

Track Reference Signals

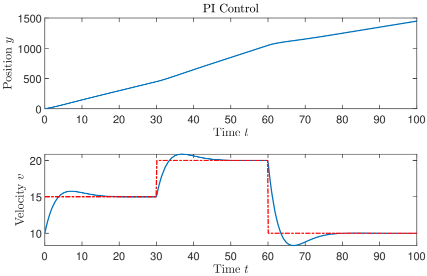
Another major application of feedback is to make a system output follow a reference value, which is called the **servo problem**.

- *Examples*: Cruise control, steering of a car, and tracking a satellite with an antenna or a star with a telescope
- *Other examples*: high performance audio amplifiers, machine tools, and industrial robots.

Cruise control. Condition: $v_0 = 10m/s$, $m = 500\text{kg}, \delta = 0.5$;
PI controller: $K_\text{p} = 250,\ K_\text{i} = 50$

Case 1: Flat road ($\theta = 0$)
Piece-wise constant desired velocity signal

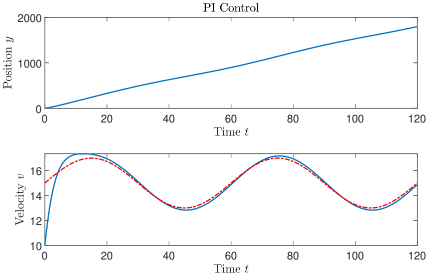
$$v_\text{des} = \begin{cases} 15m/s & t \leq 30 \\ 20m/s & 30 < t \leq 60 \\ 10m/s & 60 < t \end{cases}$$



Track Reference Signals

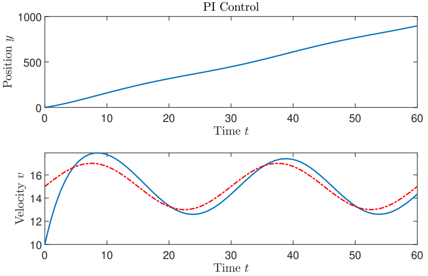
Case 2: Flat road ($\theta = 0$)
Time-varying sinusoidal signal

$$v_{\mathrm{des}} = 15 + 2 \times \sin\left(\frac{2\pi}{60}t\right)$$



Case 3: Flat road ($\theta = 0$)
Time-varying sinusoidal signal

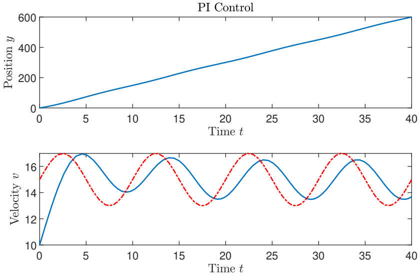
$$v_{\mathrm{des}} = 15 + 2 \times \sin\left(\frac{2\pi}{30}t\right)$$



Track Reference Signals

Case 3: Flat road ($\theta = 0$)
Time-varying sinusoidal signal

$$v_{\text{des}} = 15 + 2 \times \sin\left(\frac{2\pi}{10}t\right)$$



- To analyze and quantify the tracking behavior with respect to the frequency of the reference signal, we need to study **transfer function** representations – *bandwidth* of the closed-loop system
  - **Bandwidth**: The upper bound of the frequency of reference signals that can be tracked with small error.

Reference tracking: The same PI controller can make the closed-loop system follow a reference signal with a small tracking error.

Outline

Reduce effects of parameter variations

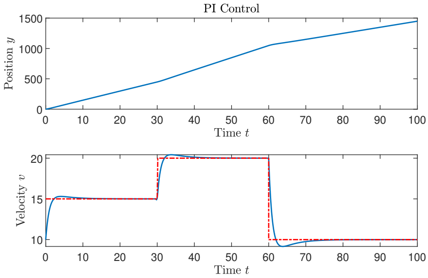
Feedback can also be used to make good systems from imprecise components (with some limitations)!

- We consider a simpler scenario, where some system parameters have variations (imprecise measurement).

Cruise control. Condition: $v_0 = 10m/s$, $m = 500\text{kg}, \delta = 0.5$;
PI controller: $K_\text{p} = 250,\ K_\text{i} = 50$

Case 1: Mass change - $m = 200kg$
Flat road ($\theta = 0$)
Piece-wise constant desired velocity signal

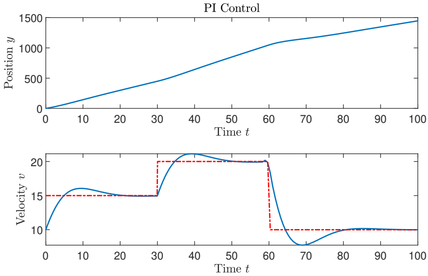
$$v_\text{des} = \begin{cases} 15m/s & t \leq 30 \\ 20m/s & 30 < t \leq 60 \\ 10m/s & 60 < t \end{cases}$$



Reduce effects of parameter variations

Case 2: Mass change - $m = 800kg$
Flat road ($\theta = 0$)
Piece-wise constant desired velocity signal

$$v_{\text{des}} = \begin{cases} 15m/s & t \leq 30 \\ 20m/s & 30 < t \leq 60 \\ 10m/s & 60 < t \end{cases}$$



Robustness: The same PI controller can make the closed-loop system follow a reference signal even when some system parameters are not known exactly.

Outline

Summary

We have used two simple examples

- A simple static model
- A simple dynamical model - Cruise control

to illustrate several fundamental properties of feedback

- Disturbance attenuation
- Reference signal tracking
- Robustness to uncertainty
- Shaping of dynamical behavior

More quantitative analysis and design techniques will be discussed later and throughout this class!