ECE 171A: Linear Control System Theory
Lecture 4: System Modeling (I)

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Reading materials: Ch 3.1, 3.2 (Advanced materials – not required)

Announcements

- HW1 due by this Friday;
- **No late policy**; Start each homework early
- Discussion this Wednesday: **2nd-order ODE and ode45**
- Midterm (I) — in class, April 26 (Week 4)
- **ECE171A Refresher - Math pre-requisites** on Canvas
  - covers some of the math concepts required in this class;
  - If you cannot finish them comfortably, you need to spend more time offline reviewing necessary backgrounds by yourself; otherwise, you may not get the desired outcomes in this course.

System modeling

A model is a *mathematical representation* of a physical, biological, or information system.

- Models allow us to reason about a system and make predictions about how a system will behave.
- The model we choose depends on the questions we wish to answer, and so there may be multiple models for a single dynamical system.
- It is important to keep in mind that all models are an approximation of the underlying system.

We here focus on two commonly used methods in feedback and control systems (**state-space domain**)

- Differential equations;
- Difference equations.

From week 4/5, we will introduce **frequency-domain** models (transfer functions).

Outline

Outline

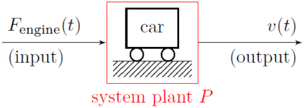
Simple examples

More examples

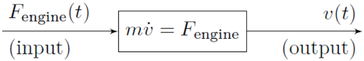
Modeling properties

Speed control

$\Rightarrow$



- **Model 1:** flat road, no friction, no air-drag



  the equation $m\dot{v} = F_{\text{engine}}$ is the system model.

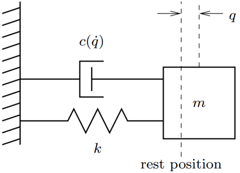
- **Model 2**: uphill with slop θ, no friction, no air-drag

$$m\dot{v} = F_{\text{engine}} - mg\sin\theta$$

- **Model 3**: uphill with slop θ, no friction, with air-drag $F_{\text{a}} = \frac{1}{2}\rho C_{\text{d}} A v^2$

$$m\dot{v} = F_{\text{engine}} - mg\sin\theta - \frac{1}{2}\rho C_{\text{d}} A v^2$$

Spring-mass system



$m =$ mass

$F =$ External force

$c =$ friction (damper)

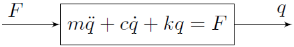
$k =$ spring stiffness

$q =$ rest position

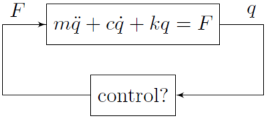
- **System model**: find the relation between the force F and the position q

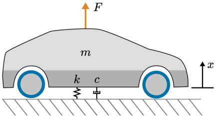
$$m\ddot{q} + c\dot{q} + kq = F.$$

- **Block diagram**



- **Feedback control**: maintain a desired position q^* with small oscillation





Modeling terminology - the control view

- When control theory emerged as a discipline in the 1940s, the modeling approach was strongly influenced by *the input/output view (e.g., transfer functions)* in electrical engineering.
- In the late 1950s, a second wave of control developments was inspired by mechanics, using *the state-space perspective*.

Standard state-space form in control (a system of first-order ODEs)

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \end{cases} \quad \leftrightarrow \quad \text{state space model}$$

- where $x(t) \in \mathbb{R}^n$ is a *state vector*, $u(t) \in \mathbb{R}^p$ is a *control variable*, and $y(t) \in \mathbb{R}^q$ is a *measured signal*.
- The dimension n of the state vector is called the *order* of the system.
- General nonlinear systems vs linear systems

$$\begin{aligned} f &: \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n, \\ h &: \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^q \end{aligned} \qquad \text{v.s.} \qquad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

Modeling terminology - the control view

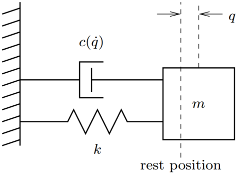
State space model

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \end{cases} \qquad \text{v.s.} \qquad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

1. **State**: capture effects of the past.
   - Consists of physical quantities that completely captures the past motion of a system for the purpose of predicting future motion.
2. **Input**: describe external excitations.
   - Inputs are extrinsic to the system dynamics (externally specified).
   - These include disturbances and control inputs.
3. **Output**: describe measured quantities.
   - Outputs are functions of the state and inputs; they are not independent variables.
4. **Dynamics**: describe state evolution.
   - Dynamics essentially take the form of update rules for the system state.

Outline

Spring-mass system revisited



System model: find the relation between the force F and the position q

$$m\ddot{q} + c\dot{q} + kq = F.$$

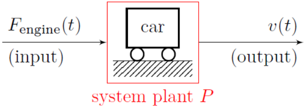
- Convert it to "standard form" (a system of first-order ODEs) by setting

$$x_1 = q,\ x_2 = \dot{q},\ y = x_1 = q,\ \text{and } u = F,$$

- So the model becomes

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{m}(-cx_2 - kx_1 + u) \end{bmatrix} \\ y = x_1. \end{cases} \Leftrightarrow \begin{cases} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \times u. \end{cases}$$

Vehicle model revisited



Model 2: uphill with slop θ, no friction, no air-drag

$$\begin{aligned} \dot{p} &= v, \\ m\dot{v} &= F_{\text{engine}} - mg\sin\theta. \end{aligned}$$

- Convert it to "standard form" (a system of first-order ODEs) by setting

$$x_1 = p,\ x_2 = \dot{p},\ y = x_2 = v, \text{ and } u = F_{\text{engine}},$$

- So the model becomes

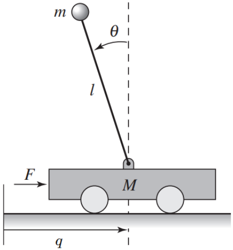
$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{m}u - g\sin\theta \end{bmatrix} \\ y = x_2. \end{cases} \Leftrightarrow \begin{cases} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{u}{m} - g\sin\theta \end{bmatrix} \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \times u. \end{cases}$$

Balance systems

A balance system is a mechanical system in which the center of mass is balanced above a pivot point .


(a) Segway


(b) Saturn rocket



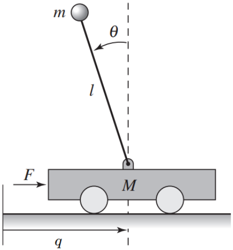
(c) Cart–pendulum system

- Generalization of the spring-mass system: **Newtonian mechanics**

$$M(q)\ddot{q} + C(q,\dot{q}) + K(q) = B(q)u,$$

where $M(q)$ is the inertia, $C(q,\dot{q})$ denotes the damping, $K(q)$ gives the forces, and $B(q)$: how the external applied forces u to the dynamics.

Cart-pendulum system



State variable:

- $q, \dot{q}$ - position and velocity of the base of the system
- $\theta, \dot{\theta}$ - angle and angular rate of the structure above the base

Output: position and angle
Control: the force F applied at the base.

$$\begin{bmatrix} M+m & -ml\cos\theta \\ -ml\cos\theta & (J+ml)^2 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} c\dot{q} + ml\sin\theta\dot{\theta}^2 \\ \gamma\dot{\theta} - mgl\sin\theta \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}.$$

Rewrite the dynamics in state space form by defining $x = (q, \theta, \dot{q}, \dot{\theta})$ and $u = F$.

$$\frac{d}{dt}\begin{bmatrix} q \\ \theta \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ \dot{\theta} \\ \dfrac{-mls_\theta\dot{\theta}^2 + mg(ml^2/J_t)s_\theta c_\theta - c\dot{q} - (\gamma/J_t)mlc_\theta\dot{\theta} + u}{M_t - m(ml^2/J_t)c_\theta^2} \\ \dfrac{-ml^2 s_\theta c_\theta\dot{\theta}^2 + M_t gls_\theta - clc_\theta\dot{q} - \gamma(M_t/m)\dot{\theta} + lc_\theta u}{J_t(M_t/m) - m(lc_\theta)^2} \end{bmatrix}$$

where $M_t = M + m, J_t = J + ml^2, c_\theta = \cos\theta, s_\theta = \sin\theta$.

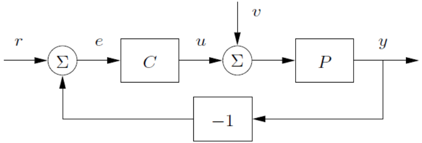
Block diagrams

A special graphical representation called a **block diagram** has been developed in control engineering.

- Emphasize the information flow and to hide details of the system.

In a block diagram, different process elements are shown as boxes:

- Each box has inputs denoted by lines with arrows pointing toward the box and outputs denoted by lines with arrows going out of the box.
- The inputs denote the variables that influence a process
- The outputs denote the signals that we are interested in or signals that influence other subsystems.



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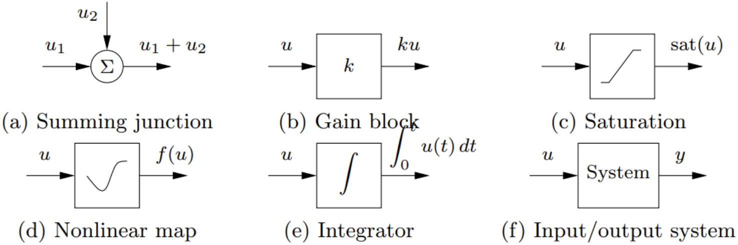

Figure: Standard block diagram elements. The arrows indicate the the inputs and outputs of each element, with the mathematical operation corresponding to the block labeled at the output.

Outline

Modeling properties

The choice of inputs and outputs depends on point of view.

- Inputs in one model might be outputs of another model (e.g., the output of a cruise controller provides the input to the vehicle model.)
- Outputs are what physical variables (often states) can be measured.
- The choice of outputs depends on what you can sense and what parts of the component model interact with other component models

The choice of state is not unique.

- There may be many choices of variables that can act as the state.
  - A trivial example: One can choose different units (scaling factors)
  - A less trivial example: One can take sums and differences of some variables.

Modeling properties

A system may be described by many different types of models:

- Ordinary differential equations
- Difference equations
- Finite-state machines for manufacturing, internet, and information flow
- Partial differential equations for fluid flow, solid mechanics, etc.
- Block diagram representation
- Black-boxes (model from experiments)

More examples will be discussed in Lecture 5; Other details can refer to *Reading materials: Ch 3.1, 3.2*.