ECE 171A: Linear Control System Theory

Lecture 6: System solutions and Phase portraits

Yang Zheng

Assistant Professor, ECE, UCSD

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Reading materials: Ch 5.1 - 5.2

Announcements

- Final reminder: HW1 due by tonight;
- HW2 will be out today, and due by next Friday.
  - **No late policy**; Start each homework early;
  - Write the number of hours (including reading lecture notes and/or textbook) on the front page for each assignment.
- Another TA: Brady Liu (Graduate Student, ECE)
- **Office hours**
  - Tuesdays, 6:30 pm - 8:30 pm (Yang Zheng, FAH 3002)
  - Thursdays, 6:30 pm - 8:30 pm (Rich Pai, FAH 3002)
  - Fridays, 6:30 pm - 8:30 pm (Brady Liu, FAH 3002)

Outline

Motivation

Solving differential equations

Qualitative analysis: phase portraits

Summary

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Summary

Modeling + Feedback control

A model is a *mathematical representation* of a physical, biological, or information system.

- Models allow us to reason about a system and make predictions about how a system will behave.

State-space model

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \end{cases} \qquad \text{v.s.} \qquad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

where $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n, h : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^q$, and $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{q \times n}$, $D \in \mathbb{R}^{q \times p}$

- $q > 1, p > 1$: *Multiple-input Multiple-output (MIMO) systems*
- $q = 1, p = 1$ (but $n \geq 1$): *Single-input Single-output (SISO) systems* $\to$ the main focus of this class.

Modeling + Feedback Control

Control goal: designing the input $u(t)$ such that

- $x(t) \to 0$ (*Regulation problem*)
- $x(t) \to x_{\mathrm{des}}(t)$ or $y(t) \to y_{\mathrm{des}}(t)$ (*Servo problem*)
- Transient behavior – quick response, less overshot/oscillation, etc
- Robustness (uncertainty, parameter variations), disturbance rejection, etc.

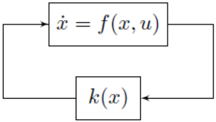
We begin by investigating systems in which the input has been set to a function of the state, e.g., $u = k(x)$

- This is one of the simplest types of feedback
- The system regulates its own state/behavior.

Closed-loop system:

$$\dot{x}(t) = f(x, k(x)) := F(x).$$

- Analytical or *Computational* solutions



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Solutions of dynamical systems

Consider a dynamical system (or a vector of ODEs)

$$\dot{x} = F(x) \tag{1}$$

where $x \in \mathbb{R}^n, F : \mathbb{R}^n \to \mathbb{R}^n$.

- We say $x(t)$ is a solution of (1) on the time interval $[t_0, t_\text{f}]$ if

$$\frac{d}{dt}x(t) = F(x(t)), \quad \forall t_0 < t < t_\text{f}$$

- **Initial value problem**: we say $x(t)$ is a solution of (1) with initial value $x_0 \in \mathbb{R}^n$ at $t_0 \in \mathbb{R}$ if

$$x(t_0) = x_0 \quad \text{and} \quad \frac{d}{dt}x(t) = F(x(t)), \quad \forall t_0 < t < t_\text{f}$$

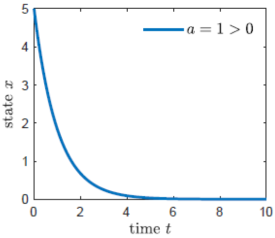
- For most differential equations we will see, there is a *unique* solution with a given initial value $x_0 \in \mathbb{R}^n$.

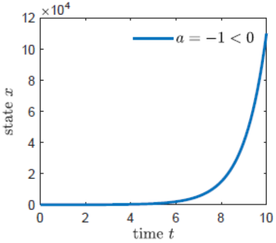
Example 1: system in one scalar variable

We have seen the following differential equation in Lecture 2 (as well as in discussion sessions and HW1)

$$\dot{x} = -ax, \qquad \text{with } x \in \mathbb{R}$$

- Its solution is $x(t) = e^{-at}x(0)$.





Example 2: simple two-dimensional systems

Consider the following decoupled/diagonal two-dimensional system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -ax_1 \\ -bx_2 \end{bmatrix} \quad \Longleftrightarrow \quad \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- The solution is

$$x_1 = e^{-at}x_1(0), \qquad x_2 = e^{-bt}x_2(0)$$

- Recall the general solution to $\dot{x} = Ax$ with initial value $x(0) \in \mathbb{R}^2$ is

$$\begin{aligned} x(t) = e^{At}x(0) &= \begin{bmatrix} e^{-at} & 0 \\ 0 & e^{-bt} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ &= \begin{bmatrix} e^{-at}x_1(0) \\ e^{-bt}x_2(0) \end{bmatrix} \end{aligned}$$

Example 3: double integrator

Consider the following dynamical system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= a \end{aligned} \quad \Longleftrightarrow \quad \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ a \end{bmatrix}$$

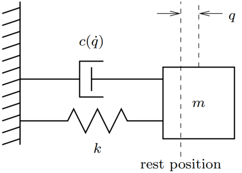
where $a \in \mathbb{R}$ is a constant.

- Its solution is

$$\begin{aligned} x_1(t) &= \frac{1}{2}at^2 + x_2(0)t + x_1(0) \\ x_2(t) &= at + x_2(0) \end{aligned}$$

- Consider the system state as position $x_1(t)$ + velocity $x_2(t)$.

Example 4: Damped oscillator (spring-mass)



$m =$ mass

$F =$ External force

$c =$ friction (damper)

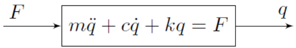
$k =$ spring stiffness

$q =$ rest position

- **System model**: find the relation between the force F and the position q

$$m\ddot{q} + c\dot{q} + kq = F.$$

- **Block diagram**



- **Free response:** Let $F = 0$, we have

$$m\ddot{q} + c\dot{q} + kq = 0 \quad \Rightarrow \quad \ddot{q} + \frac{c}{m}\dot{q} + \frac{k}{m}q = 0.$$

- Introduce $\zeta \in \mathbb{R}, \omega_0 \in \mathbb{R}$ such that

$$2\zeta\omega_0 = \frac{c}{m}, \omega_0^2 = \frac{k}{m} \quad \Rightarrow \quad \ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = 0.$$

Example 4: Damped oscillator (spring-mass)

- We set

$$x_1 = q, \qquad x_2 = \frac{\dot{q}}{\omega_0}$$

- Then, we have the **standard state-space form**

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \omega_0 x_2 \\ -\omega_0 x_1 - 2\zeta\omega_0 x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & -2\zeta\omega_0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Upon denoting $w_{\mathrm{d}} = \omega_0\sqrt{1-\zeta^2}$ (we have assumed that $\zeta < 1$ – "underdamped" oscillator), its solution is in the form of

$$x_1(t) = e^{-\zeta\omega_0 t}\left(a_1 \cos(\omega_{\mathrm{d}} t) + b_1 \sin(\omega_{\mathrm{d}} t)\right)$$
$$x_2(t) = e^{-\zeta\omega_0 t}\left(a_2 \cos(\omega_{\mathrm{d}} t) + b_2 \sin(\omega_{\mathrm{d}} t)\right)$$

where a_1, a_2, b_1, b_2 are constants depending on initial conditions $x_1(0), x_2(0)$
(their values can be found in Example 5.1 in the textbook).

Example 4: Damped oscillator (spring-mass)

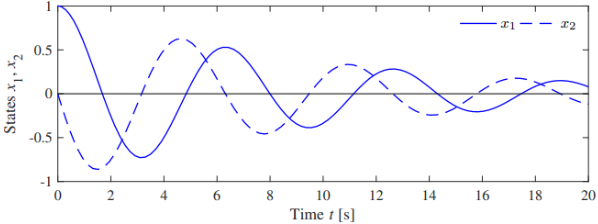


Figure 5.1: Response of the damped oscillator to the initial condition $x_0 = (1, 0)$. The solution is unique for the given initial conditions and consists of an oscillatory solution for each state, with an exponentially decaying magnitude.

Existence and uniqueness of solutions

- Recall that an n**th-order linear ODE**

$$\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_1\frac{d}{dt}y(t) + a_0 y(t) = u(t)$$

- with with initial values

$$y(t_0) = y_0, \quad \dot{y}(t_0) = y_1, \quad \ldots \quad y^{(n-1)}(t_0) = y_{n-1}.$$

Theorem

Let $u(t)$ be a continuous function on an interval $\mathcal{I} = [t_1, t_2]$. Then, for any $t_0 \in \mathcal{I}$, a solution $y(t)$ of the initial value problem exists on $\mathcal{I}$ and is unique.

The following (possibly nonlinear) differential equation

$$\dot{x} = F(x)$$

where $x \in \mathbb{R}^n, F : \mathbb{R}^n \to \mathbb{R}^n$,

- *may not have a solution for all t* (see Example 5.2: $\dot{x} = x^2$)
- *may not have a unique solution* (see Example 5.3: $\dot{x} = 2\sqrt{x}$)
- Focus on linear ODEs; for time-domain simulations, ode45 is your friend.

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Qualitative analysis: phase portraits

Summary

Time plot vs. Phase portrait

The state evolution of dynamical systems can be described using either a *time plot* or a *phase portrait*.

- **Time plot**: shows the values of the individual states as a function of time
- **Phase portrait**: illustrates how the states move in the state space; gives a strong intuitive representation of the equation as a vector field/a flow.

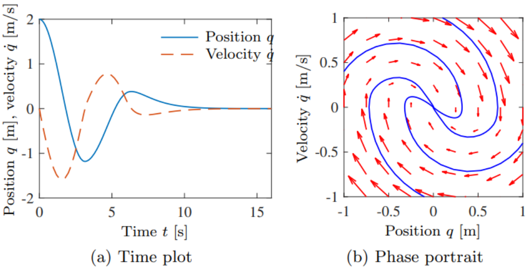

Figure 3.2: Illustration of a state model. A state model gives the rate of change of the state as a function of the state. The plot on the left shows the evolution of the state as a function of time. The plot on the right, called a *phase portrait*, shows the evolution of the states relative to each other, with the velocity of the state denoted by arrows.

Phase portraits

Planar dynamical systems: two state variables $x \in \mathbb{R}^2$, allowing their solutions to be plotted in the (x_1, x_2) plane.

- **Vector field**: consider a system of ODEs $\dot{x} = F(x)$. The right-hand side defines a *velocity* $F(x) \in \mathbb{R}^n$ at every $x \in \mathbb{R}^n$.
  It shows how x changes and can be represented as a vector $F(x) \in \mathbb{R}^n$.
- **Phase portrait**: shows the evolution of the states from different initial conditions: it illustrates *how the states move* in the state space.

Damped oscillator

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = 0$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & -2\zeta\omega_0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

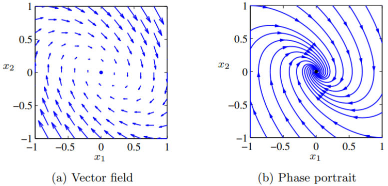

Figure 5.3: Phase portraits. (a) This plot shows the vector field for a planar dynamical system. Each arrow shows the velocity at that point in the state space. (b) This plot includes the solutions (sometimes called streamlines) from different initial conditions, with the vector field superimposed.

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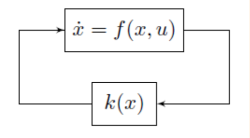
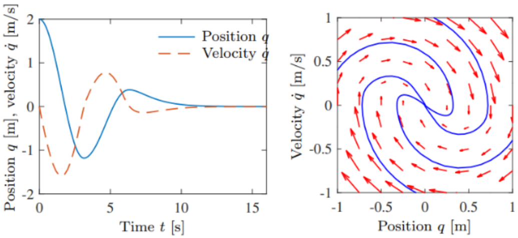
Analytical or *Computational* solutions



- **Solving differential equations**
- **Qualitative analysis**: phase portraits and time plot



(a) Time plot (b) Phase portrait