ECE 171A: Linear Control System Theory

Lecture 8: Linearization

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Reading materials: Ch 6.1, Ch 6.4

Outline

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Stability and solutions of linear systems

Theorem (Stability of a linear system)

The system $\dot{x} = Ax$ *is*

- **asymptotically stable** *if and only if all eigenvalues of* A *have a strictly negative real part, i.e.,*

$$\mathrm{Re}(\lambda_i) < 0, i = 1, \ldots, n.$$

- **unstable** *if any eigenvalues* A *has a strictly positive real part, i.e, there exist* i *such that*

$$\mathrm{Re}(\lambda_i) > 0.$$

The case with $\mathrm{Re}(\lambda_i) \leq 0$ is more difficult, which is beyond the scope of this class; see the example in Lecture 7.

The general solution of $\dot{x} = Ax$ with initial state $x(0) \in \mathbb{R}^n$ is

$$x(t) = e^{At} x(0).$$

Approximation of nonlinear systems

In practice, almost all physical systems are not linear (i.e., nonlinear)

- No control input

$$\dot{x} = F(x)$$

- With control input

$$\dot{x} = f(x, u)$$

Common practice:

- Approximate a nonlinear system by a linear one;
- Design controllers based on an approximate linear model;
- Verify the results by simulating the closed-loop system using a nonlinear model.

Taylor series

The **Taylor series** of a real function $f(x)$ that is infinitely differentiable at a real number a is the power series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

(*if the sum/series converges*)

Example

- Exponential function e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

- Trigonometric functions: $\sin x$ and $\cos x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

Outline

Motivation

Jacobian Linearization (I): no control input

Jacobian Linearization (II): with control input

Summary

Nonlinear systems

Suppose that we have a nonlinear system

$$\dot{x} = F(x),$$

that has an equilibrium point at x_{e}.

- Compute the Taylor series expansion of the vector field

$$F(x) = F(x_{\mathrm{e}}) + \left.\frac{\partial F}{\partial x}\right|_{x_{\mathrm{e}}} (x - x_{\mathrm{e}}) + \text{higher-order terms in } (x - x_{\mathrm{e}}).$$

- Since we have $F(x_{\mathrm{e}}) = 0$, we have

$$\dot{x} = \left.\frac{\partial F}{\partial x}\right|_{x_{\mathrm{e}}} (x - x_{\mathrm{e}}) + \text{higher-order terms in } (x - x_{\mathrm{e}}).$$

- Choose a new state variable $z = x - x_{\mathrm{e}}$, and we can approximate the system as

$$\dot{z} = Az, \qquad \text{with} \quad A = \left.\frac{\partial F}{\partial x}\right|_{x_{\mathrm{e}}}.$$

Example: Inverted pendulum

Example

Consider a damped inverted pendulum with open-loop dynamics as

$$\dot{x} = \begin{bmatrix} x_2 \\ \sin x_1 - c x_2 \end{bmatrix}, \qquad \text{where } \; x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}.$$

- **Step 1**: find equilibrium points

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \pi \\ 0 \end{bmatrix},$$

- **Step 2**: Linearize the system around $(0, 0)$

$$\begin{aligned} f_1(x_1, x_2) &= x_2 \\ f_2(x_1, x_2) &= \sin x_1 - c x_2 \approx f_2(0,0) + \left.\frac{\partial f_2}{\partial x_1}\right|_{(0,0)} (x_1 - 0) + \left.\frac{\partial f_2}{\partial x_2}\right|_{(0,0)} (x_2 - 0) \\ &= 0 + x_1 - c x_2 \end{aligned}$$

- **Step 3**: get a linear model $\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & -c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Example: Inverted pendulum

Example

Consider an inverted pendulum with open-loop dynamics as

$$\dot{x} = \begin{bmatrix} x_2 \\ \sin x_1 - c x_2 \end{bmatrix}, \qquad \text{where } x = \left[\theta, \dot{\theta}\right]^\top .$$

- **Step 2**: Linearize the system around $(\pi, 0)$

$$\begin{aligned} f_1(x_1, x_2) &= x_2 \\ f_2(x_1, x_2) &= \sin x_1 - c x_2 \approx f_2(\pi, 0) + \left.\frac{\partial f_2}{\partial x_1}\right|_{(\pi,0)} (x_1 - \pi) + \left.\frac{\partial f_2}{\partial x_2}\right|_{(\pi,0)} (x_2 - 0) \\ &= 0 - 1 \times (x_1 - \pi) - c x_2 \end{aligned}$$

- **Step 3**: Define a new state variable $z_1 = x_1 - \pi$ and $z_2 = x_2$

$$\dot{z} = \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Lyapunov's first (Indirect) method

Theorem

Consider a nonlinear system $\dot{x} = F(x)$, *with the origin* $x_{\mathrm{e}} = 0$ *as an equilibrium point. Let*

$$A = \left.\frac{\partial F}{\partial x}\right|_{x_{\mathrm{e}}=0}$$

- $x_{\mathrm{e}} = 0$ *is locally asymptotically stable if* A *is asymptotically stable or all eigenvalues of* A *have negative real parts.*
- $x_{\mathrm{e}} = 0$ *is unstable if one or more of the eigenvalues of* A *has positive real part.*

- June 6, 1857 - November 3, 1918
- Russian mathematician, mechanician and physicist.
- Many important contributions in the **stability theory** of a dynamical system, **mathematical physics** and **probability theory**.

Example: Inverted pendulum

Consider an inverted pendulum with open-loop dynamics as

$$\dot{x} = \begin{bmatrix} x_2 \\ \sin x_1 - c x_2 \end{bmatrix}, \qquad \text{where } x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}.$$

- Equilibrium one $(0, 0)$: **Unstable**

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & -c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

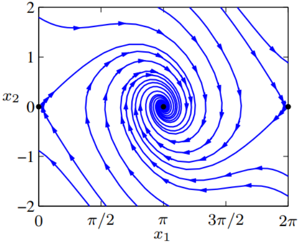
- Compute its eigenvalues

$$\left| \lambda I - \begin{bmatrix} 0 & 1 \\ 1 & -c \end{bmatrix} \right| = \left| \begin{bmatrix} \lambda & -1 \\ -1 & \lambda + c \end{bmatrix} \right| = \lambda^2 + c\lambda - 1 = 0$$

- Equilibrium two $(\pi, 0)$: **Stable**

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \pi \\ 0 \end{bmatrix} \quad \Rightarrow \quad \dot{z} = \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Example: Inverted pendulum



(a) Nonlinear model

(b) Linear approximation

Figure: Comparison between the phase portraits for the full nonlinear system (a) and its linear approximation around the origin (b). Notice that near the equilibrium point at the center of the plots, the phase portraits are almost identical.

Outline

Motivation

Jacobian Linearization (I): no control input

Jacobian Linearization (II): with control input

Summary

Nonlinear system - Linearization

- Given a nonlinear dynamical system

$$\dot{x} = f(x,u), \qquad y = h(x,u).$$

- Suppose $f(x_\mathrm{e}, u_\mathrm{e}) = 0$ for a fixed point $(x_\mathrm{e}, u_\mathrm{e})$. Let $y_\mathrm{e} = h(x_\mathrm{e}, u_\mathrm{e})$.
- Define a new set of states, inputs, and outputs

$$\tilde{x} = x - x_\mathrm{e}, \qquad \tilde{u} = u - u_\mathrm{e}, \qquad \tilde{y} = y - y_\mathrm{e}.$$

- Then, apply a Taylor series expansion

$$\begin{aligned} \frac{d\tilde{x}}{dt} = \frac{dx}{dt} &= f(x_\mathrm{e} + \tilde{x}, u_\mathrm{e} + \tilde{u}) \\ &= f(x_\mathrm{e}, u_\mathrm{e}) + \left.\frac{\partial f}{\partial x}\right|_{(x_\mathrm{e}, u_\mathrm{e})} \tilde{x} + \left.\frac{\partial f}{\partial u}\right|_{(x_\mathrm{e}, u_\mathrm{e})} \tilde{u} + \mathcal{O}(\|\tilde{x}, \tilde{u}\|^2) \\ &\approx A\tilde{x} + B\tilde{u} \end{aligned}$$

where we have applied the fact $f(x_\mathrm{e}, u_\mathrm{e}) = 0$, and

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_\mathrm{e}, u_\mathrm{e})}, \qquad B = \left.\frac{\partial f}{\partial u}\right|_{(x_\mathrm{e}, u_\mathrm{e})}.$$

Nonlinear system - Linearization

- Similarly, we have

$$\begin{aligned}\tilde{y} = y - y_{\mathrm{e}} &= h(x_{\mathrm{e}} + \tilde{x}, u_{\mathrm{e}} + \tilde{u}) - h(x_{\mathrm{e}}, u_{\mathrm{e}}) \\ &\approx \left.\frac{\partial h}{\partial x}\right|_{(x_{\mathrm{e}}, u_{\mathrm{e}})} \tilde{x} + \left.\frac{\partial h}{\partial u}\right|_{(x_{\mathrm{e}}, u_{\mathrm{e}})} \tilde{u} \\ &= C\tilde{x} + D\tilde{u}\end{aligned}$$

The Jacobian linearization of the nonlinear system

$$\dot{x} = f(x, u), \qquad y = h(x, u), \tag{1}$$

at an equilibrium point $(x_{\mathrm{e}}, u_{\mathrm{e}})$ (such that $f(x_{\mathrm{e}}, u_{\mathrm{e}}) = 0$) is

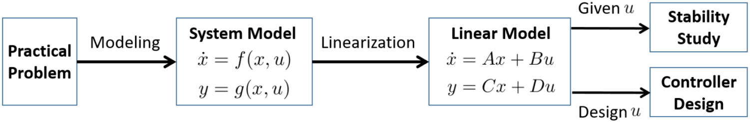
$$\frac{d\tilde{x}}{dt} = A\tilde{x} + B\tilde{u}, \qquad \tilde{y} = C\tilde{x} + D\tilde{u}, \tag{2}$$

where $\tilde{x} = x - x_{\mathrm{e}}, \qquad \tilde{u} = u - u_{\mathrm{e}}, \qquad \tilde{y} = y - y_{\mathrm{e}}$, and

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_{\mathrm{e}}, u_{\mathrm{e}})}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{(x_{\mathrm{e}}, u_{\mathrm{e}})}, \quad C = \left.\frac{\partial h}{\partial x}\right|_{(x_{\mathrm{e}}, u_{\mathrm{e}})}, \quad D = \left.\frac{\partial h}{\partial u}\right|_{(x_{\mathrm{e}}, u_{\mathrm{e}})}.$$

Summary

The linear system (2) approximates the original nonlinear system (1).



Practical Problem

Modeling

System Model
$\dot{x} = f(x,u)$
$y = g(x,u)$

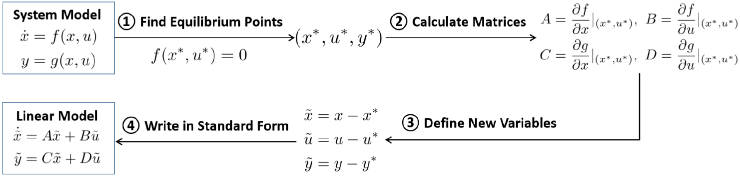
Linearization

Linear Model
$\dot{x} = Ax + Bu$
$y = Cx + Du$

Given u → **Stability Study**

Design u → **Controller Design**

Figure: General framework (taken from Prof Na Li's ES 155)



System Model
$\dot{x} = f(x,u)$
$y = g(x,u)$

① **Find Equilibrium Points**
$f(x^*,u^*) = 0$
(x^*,u^*,y^*)

② **Calculate Matrices**
$A = \frac{\partial f}{\partial x}|_{(x^*,u^*)}, \; B = \frac{\partial f}{\partial u}|_{(x^*,u^*)}$
$C = \frac{\partial g}{\partial x}|_{(x^*,u^*)}, \; D = \frac{\partial g}{\partial u}|_{(x^*,u^*)}$

③ **Define New Variables**
$\tilde{x} = x - x^*$
$\tilde{u} = u - u^*$
$\tilde{y} = y - y^*$

④ **Write in Standard Form**

Linear Model
$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$
$\tilde{y} = C\tilde{x} + D\tilde{u}$

Figure: Model Linearization Procedure (taken from Prof Na Li's ES 155)

Example: SpaceX rocket controller design

A rocket of mass m in vertical flight can be modeled by

$$\dot{h} = v$$

$$M\dot{v} = F - \frac{km}{h^2} - cv,$$

- $h > 0$ is the vertical distance away from the earth,
- v is the vertical velocity,
- F is the rocket engine thrust force (control input),
- $\frac{km}{h^2}$ represents the universal gravitation, and cv captures the friction.

Suppose $m = 1, k = 1, c = 1$; we let $x_1 = h$ and $x_2 = v$, and the output $y = h$, input $u = F$.

Question 1 - equilibrium point: Let $F^* = 1$. What is the equilibrium point of this system?

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{1}{x_1^2} - x_2 + u \end{aligned} \qquad \Longrightarrow \qquad \begin{cases} u^* = 1, x_1^* = 1, x_2^* = 0, \\ y^* = x_1^* = 1 \end{cases}$$

Example: SpaceX rocket controller design

Question 2 - Linearization: Linearize the system around the equilibrium point.

- **Step 1**: Write down the (possibly nonlinear) dynamics (step 0: obtain the equilibrium)

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, u) = x_2 \\ \dot{x}_2 = f_2(x_1, x_2, u) = -\dfrac{1}{x_1^2} - x_2 + u \end{cases}$$

- **Step 2**: compute their partial derivatives

$$\frac{\partial f_1}{\partial x_1} = 0, \quad \frac{\partial f_1}{\partial x_2} = 1, \quad \frac{\partial f_1}{\partial u} = 0,$$
$$\frac{\partial f_2}{\partial x_1} = \frac{2}{x_1^3}, \quad \frac{\partial f_2}{\partial x_2} = -1, \quad \frac{\partial f_2}{\partial u} = 1,$$

- **Step 3**: define new variables $\tilde{x} = x - x^*$, $\tilde{u} = u - u^*$, and $\tilde{y} = y - y^*$.
- **Step 4**: Finalize the linearized model

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{u}, \qquad \tilde{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}.$$

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Consider a nonlinear system $\dot{x} = F(x)$, with $x_{\mathrm{e}} = 0$ as an equilibrium point. Let

$$A = \left.\frac{\partial F}{\partial x}\right|_{x_{\mathrm{e}}=0}$$

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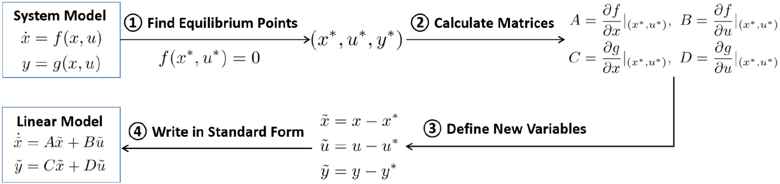

System Model

$\dot{x} = f(x, u)$

$y = g(x, u)$

① Find Equilibrium Points

$f(x^*, u^*) = 0$

(x^*, u^*, y^*)

② Calculate Matrices

$A = \frac{\partial f}{\partial x}|_{(x^*,u^*)}, \quad B = \frac{\partial f}{\partial u}|_{(x^*,u^*)}$

$C = \frac{\partial g}{\partial x}|_{(x^*,u^*)}, \quad D = \frac{\partial g}{\partial u}|_{(x^*,u^*)}$

③ Define New Variables

$\tilde{x} = x - x^*$

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$\tilde{y} = y - y^*$

④ Write in Standard Form

Linear Model

$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$

$\tilde{y} = C\tilde{x} + D\tilde{u}$

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