ECE285: Semidefinite and sum-of-squares optimization **Winter 2023**

Homework 1

Date Given: January 12, 2023
Date Due: January 20, 11:59 pm PST

Note: *Please write down the number of the hours that you spend on the homework assignment.*

Problem 1: Properties of positive semidefinite matrices [30 pts]

Classify the following statements as true or false. A proof or counterexample is required (all matrices below have compatible dimensions) [5pts each].

(a) The diagonal elements of a positive definite matrix are all positive. The diagonal elements of a positive semidefinite matrix are all nonnegative.

(b) A symmetric matrix $Q \in \mathbb{S}^n$ of rank r is positive semidefinite if and only if there exists a square matrix $R \in \mathbb{R}^{n \times n}$ of rank r such that $Q = RR^{\mathsf{T}}$. If Q is positive definite, then R is non-singular.

(*Hint: consider the spectral decomposition of symmetric matrices* $Q = U\Lambda U^{\mathsf{T}}$ *with* $UU^{\mathsf{T}} = I$.)

(c) If $A \succeq 0, B \succeq 0$ then $\langle A, B \rangle \geq 0$, and further $\langle A, B \rangle = 0$ implies $AB = 0$. If $A \succeq 0, B \succeq 0$ and $\langle A, B \rangle = 0$, then $A = 0$ or $B = 0$.

(d) Consider a symmetric matrix $A \in \mathbb{S}^n$. If $\langle A, B \rangle \geq 0, \forall B \succeq 0$, then A is positive semidefinite.

(e) If $A \succeq 0$ and $\text{trace}(A) = 0$, then $A = 0$.

(f) If $A \succeq 0, B \succeq 0$, and $A + B = 0$, then $A = B = 0$.

Problem 2: Norms, dual norms, and induced norms [25 pts]

Establish the following statements [5pts each]

(a) Consider $x \in \mathbb{R}^2$. Prove that $f(x) = \sqrt{2x_1^2 - 2x_1x_2 + x_2^2}$ is a norm in $\mathbb{R}^2$.

(b) Let $Q \in \mathbb{S}^n_{++}$. Prove that $f(x) = \sqrt{x^{\mathsf{T}}Qx}$ is a norm in $\mathbb{R}^n$.

(c) Let $Q \in \mathbb{S}^n_{++}$. Prove that Q^{-1} exists and it is positive definite too. Show that the dual norm of $f(x) = \sqrt{x^{\mathsf{T}}Qx}$ is given by

$$g(x) = \sqrt{x^{\mathsf{T}}Q^{-1}x}.$$

Hint: you may want to use the factorization $Q = RR^{\mathsf{T}}$. *You first need to prove this factorization exists if you do. You may also want to use the fact that (the dual norm of* $\|\cdot\|_2$ *in* $\mathbb{R}^n$ *is itself)*

$$\max_{\|u\|_2 \leq 1} v^{\mathsf{T}}u = \|v\|_2.$$

Note that by the generalized Cauchy-Schwartz inequality, we have

$$|x^{\mathsf{T}}y| \leq \sqrt{x^{\mathsf{T}}Qx}\sqrt{y^{\mathsf{T}}Q^{-1}y}, \forall x, y \in \mathbb{R}^n.$$

(d) Let $A \in \mathbb{R}^{m \times n}$. Prove that its induced 2 norm, also known as, spectral norm, is given by

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^{\mathsf{T}}A)}.$$

(e) Let $A \in \mathbb{R}^{m\times n}$. Prove the following inequality

$$\|A\|_2 \le \|A\|_{\mathrm{F}} \le \|A\|_{2*},$$

where $\|A\|_2$ denotes the spectral norm, $\|A\|_{\mathrm{F}}$ denotes the Frobenius norm, and $\|A\|_{2*} := \sum_{i=1}^{r} \sigma_i(A)$ denotes the nuclear norm.

Hint: this can be viewed as a generalization for vector norms

$$\|u\|_\infty \le \|u\|_2 \le \|u\|_1, \qquad \forall u \in \mathbb{R}^n.$$

Problem 3: Singular value decomposition and image compression [45 pts]

Let $A \in \mathbb{R}^{m\times n}$ of rank r. Consider the singular value decomposition (SVD) of matrices $A = U\Sigma V^\mathsf{T}$ with $U^\mathsf{T}U = I_m$, $V^\mathsf{T}V = I_n$ and $\Sigma \in \mathbb{R}^{m\times n}$ including r singular values $\sigma_1, \cdots, \sigma_r$ on the diagonal of its upper left $r \times r$ block and zeros everywhere else. These singular values are given by

$$\sigma_i = \sqrt{i\text{-th eigenvalue of } A^\mathsf{T}A},$$

and in general they appear in descending order:

$$\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r.$$

The columns of U and V are respectively called the left and right singular vectors of A and can be obtained by taking an orthonormal set of eigenvectors for the matrices AA^T and $A^\mathsf{T}A$. In Matlab, the command `svd` can output the three matrices U, Σ, and V.

For a positive integer $k \le \min\{m, n\}$, we let A_k denote an $m \times n$ matrix which is an "approximation" of the matrix A obtained from its top k singular values and singular vectors. Formally, we have

$$A_k := U_k \Sigma_k V_k^\mathsf{T}, \tag{1}$$

where U_k has the first k columns of U, V_k has the first k columns of V, and Σ_k is the upper left $k \times k$ block of Σ. Now consider an optimization problem:

$$\min_{B\in\mathbb{R}^{m\times n}, \mathrm{rank}(B)\le k} \|A - B\|. \tag{2}$$

where $\|.\|$ denotes a suitable matrix norm. Establish the following statements.

(a) Let $A \in \mathbb{R}^{m\times n}$. Prove that $A^\mathsf{T}A$ is positive semidefinite. [5 pts]

(b) Prove that if A is symmetric, then the singular values of A are the same as the absolute value of the eigenvalues of A. [5 pts]

(c) Consider this optimization problem (2) for the spectral norm $\|.\|_2$. (Recall that the spectral norm of a matrix is defined as $\|C\|_2 = \max_{\|x\|_2=1} \|Cx\|_2$.) Prove that the matrix A_k, defined in (1), is an optimal solution to (2). [10 pts]

Hint: SVD allows you to write $A = \sum_{i=1}^{r} \sigma_i u_i v_i^\mathsf{T}$ (rank-1 decomposition), where r is the rank of A. If we let $\sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_r$, we have $A_k = \sigma_1 u_1 v_1^\mathsf{T} + \ldots + \sigma_k u_k v_k^\mathsf{T}$. Then, you may want to prove

- *The spectral norm of the approximation error $A - A_k$ is $\|A - A_k\| = \sigma_{k+1}$.*
- *For any matrix B of rank at most k, we have $\|A - B\| \ge \sigma_{k+1}$. For this, you may also want to use the fact that for any matrix $E \in \mathbb{R}^{m\times n}$, rank(E)+dim null(E) $= n$.*

(d) Download the file `Geisel Library.jpg` into your Matlab path. You can read this in by typing:

```
A=imread('Geisel Library.jpg');
A=im2double(A);
A=rgb2gray(A);
```

Consider this optimization problem (2) for Frobenius norm $\|.\|_F$ (Recall that the Frobenius norm of a matrix is defined as $\|C\|_F = \sqrt{\sum_{i,j} C_{i,j}^2}$). For $k = 10, 40, 80, 160$, use Matlab to compute A_k as defined in (1) and output the value of $\|A - A_k\|_F$. [10 pts]

(*Include your code for this part and the next.*)

(e) Use the commands `subplot` and `imshow` to generate the original image (A) and the compressed images ($A_{10}, A_{40}, A_{80},$ and A_{160}) on the same figure frame. In addition, generate two plots demonstrating (*i*) $\|A - A_k\|_F$ versus k, and (*ii*) "total savings" versus k. Total savings is to be interpreted as the answer to the question: How many elements will you save if you store A_k instead of A? (Consider only gray-scale images) Explain why this number is equal to $mn - (n + m + 1)k$. How much are you saving for $k = 160$? [10 pts]

(f) Use the Matlab function `imwrite` to create two images A and A_{160} and show them using `imshow`. Can you tell them apart? [5 pts]