ECE285: Semidefinite and sum-of-squares optimization **Winter 2024**

Lecture 11: Nonconvex QCQP, SDP relaxations, and S-lemma

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Learning goals:

1. Nonconvex QCQP
2. S-lemma
3. Shor's semidefinite relaxation

1 Nonconvex QCQP

In this lecture, we consider semidefinite relaxation for general nonconvex quadratically constrained quadratic programming (QCQP). The problem of (nonconvex) QCQP is an optimization problem of the from

$$\begin{aligned} \min_{x} \quad & x^\mathsf{T} Q_0 x + q_0^\mathsf{T} x + c_0 \\ \text{subject to} \quad & x^\mathsf{T} Q_i x + q_i^\mathsf{T} x + c_i \leq 0, i = 1, \ldots, m \end{aligned} \tag{1}$$

where $Q_0, Q_i \in \mathbb{S}^n, q_0, q_i \in \mathbb{R}^n, c_0, c_i \in \mathbb{R}$. Here, we do not assume Q, Q_i to be positive semidefinite (if so, there exists an exact reformulation into a second-order cone program; see Lecture 6). It is not difficult to see that problem (1) is non-convex and contains the max-cut and independent set problems (see Lecture 10) as special cases. Thus, (1) is numerically hard to solve exactly.

Remark 11.1. *A equality constraint on a quadratic function $h(x) = 0$ can be modelled as two quadratic inequality constraints $h(x) \leq 0$ and $-h(x) \leq 0$. Thus,* (1) *can also include equality constraints.* □

The goal of this lecture is to introduce a powerful semidefinite relaxation for (1) that provides non-trivial lower bounds. A well-known special case that contains only one inequality constraint has an exact SDP formulation – this is known as the S-lemma. We first discuss this famous S-lemma in Section Section 2.

2 S-lemma

Here, we consider a special case of (1) with $m = 1$. For simplicity, we write

$$\begin{aligned} \min_{x} \quad & f(x) \\ \text{subject to} \quad & g(x) \leq 0, \end{aligned} \tag{2}$$

where $f(x) = x^\mathsf{T} Q_0 x + q_0^\mathsf{T} x + c_0$ and $g(x) = x^\mathsf{T} Q_1 x + q_1^\mathsf{T} x + c_1$. This problem appears in many applications, such as the trust region problem in nonlinear programming, and robust second-order cone programming.

We will show that (2) can be solved exactly via an SDP. The key is the following celebrated S-lemma.

Theorem 11.1. *Given two quadratic functions $f(x), g(x)$. Suppose $\exists x_0 \in \mathbb{R}^n$ such that $g(x_0) > 0$, then*

$$\{x \in \mathbb{R}^n \mid g(x) \geq 0\} \subseteq \{x \in \mathbb{R}^n \mid f(x) \geq 0\}$$

if and only if there exist $\lambda \geq 0$, such that $f(x) \geq \lambda g(x), \forall x \in \mathbb{R}^n$.

The "If" part is obvious, and it is a certificate for the first implication. We will prove the "only if" part by strong duality of SDPs. For those who are interested in the history and applications of S-lemma, please refer to the excellent survey [4].

2.1 SDP reformulation

We now apply the S-lemma to formulate (2) into an equivalent SDP. We assume that (2) is strictly feasible, i.e., $\exists x_0$ such that $g(x_0) < 0$. We note that

$$\begin{array}{rl} \min\limits_{x} & f(x) \\ \text{subject to} & -g(x) \geq 0, \end{array} \quad \Leftrightarrow \quad \begin{array}{rl} \max\limits_{\gamma} & \gamma \\ \text{subject to} & f(x) \geq \gamma,\ \forall x \in \{x \in \mathbb{R}^n \mid -g(x) \geq 0\}. \end{array}$$

Applying the S-lemma leads to an equivalent formulation

$$\begin{array}{rl} \max\limits_{\gamma,\lambda} & \gamma \\ \text{subject to} & f(x) - \gamma \geq -\lambda g(x), \forall x \in \mathbb{R}^n, \\ & \lambda \geq 0 \end{array} \tag{3}$$

Considering the expression of $f(x), g(x)$, it is easy to see that

$$f(x) + \lambda g(x) - \gamma = x^\mathsf{T}(Q_0 + \lambda Q_1)x + (q_0 + \lambda q_1)^\mathsf{T} x + c_0 + \lambda c_1 - \gamma$$

We have the following simple lemma.

Lemma 11.1. *A quadratic inequality with a symmetric $n \times n$ matrix A satisfies*

$$h(x) = x^\mathsf{T} A x + 2b^\mathsf{T} x + c \geq 0, \forall x \in \mathbb{R}^n$$

if and only if

$$\begin{bmatrix} c & b^\mathsf{T} \\ b & A \end{bmatrix} \succeq 0.$$

Applying Lemma 11.1, we derive the equivalent SDP formulation of (3)

$$\begin{array}{rl} \max\limits_{\gamma,\lambda} & \gamma \\ \text{subject to} & \begin{bmatrix} c_0 + \lambda c_1 - \gamma & \frac{1}{2}(q_0 + \lambda q_1)^\mathsf{T} \\ \frac{1}{2}(q_0 + \lambda q_1) & Q_0 + \lambda Q_1 \end{bmatrix} \succeq 0, \\ & \lambda \geq 0. \end{array} \tag{4}$$

Proof of Lemma 11.1: The inhomogeneous quadratic function $h(x)$ is nonnegative globally if and only if the homogeneous quadratic function

$$l(x,t) = x^\mathsf{T} A x + 2tb^\mathsf{T} x + ct^2 \geq 0, \quad \forall (x,t) \in \mathbb{R}^{n+1}. \tag{5}$$

Indeed, we have

- If $l(x,t)$ is nonnegative globally, then $h(x) = l(x,1)$ must be so.

- If $t \neq 0$, then $l(x,t) = t^2 h(\frac{x}{t}) \geq 0, \forall x \in \mathbb{R}^n$. Thus, $l(x,t)$ is nonnegative for $t \neq 0$. By continuity, $l(x,t)$ is nonnegative everywhere, $\forall (x,t) \in \mathbb{R}^{n+1}$.

Now, we have

$$l(x,t) = \begin{bmatrix} t \\ x \end{bmatrix}^\mathsf{T} \begin{bmatrix} c & b^\mathsf{T} \\ b & A \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix} \geq 0, \forall \begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R}^{n+1} \quad \Leftrightarrow \quad \begin{bmatrix} c & b^\mathsf{T} \\ b & A \end{bmatrix} \succeq 0.$$

This completes the proof.

Remark 11.2. *The homogenization argument* (5) *is very useful. We will use it later in this lecture, as well as in later lectures on sum-of-squares optimization.* □

2.2 Regularity assumption

The regularity assumption: $\exists x_0 \in \mathbb{R}^n, g(x_0) > 0$ cannot be removed in general. It is like the Slater's condition for strong duality of convex optimization problems. The following is a counterexample for S-lemma when the regularity assumption fails.

Example 11.1. *Consider $g(x) = -x^2$ and $f(x) = -x^2 + x$. Then we have*

$$\{0\} = \{x \in \mathbb{R}^n \mid g(x) \geq 0\} \subseteq \{x \in \mathbb{R}^n \mid f(x) \geq 0\}.$$

However, there exists no $\lambda \geq 0$ such that $f(x) \geq \lambda g(x), \forall x \in \mathbb{R}$. Indeed, the following inequality

$$-x^2 + x \geq -\lambda x^2$$

cannot hold globally since the linear term will dominate the quadratic term when x is close to zero, and the inequality fails. □

In fact, it is not difficult to check the regularity condition. It is equivalent to say that the following symmetric matrix

$$\begin{bmatrix} c_1 & \frac{1}{2}q_1^\mathsf{T} \\ \frac{1}{2}q_1 & Q_1 \end{bmatrix}$$

has at least one strictly positive eigenvalue.

Theorem of strong alternatives: The S-lemma tells us exactly one of the following statements is true (assume the regularity condition holds)

1. $g(x) \geq 0, f(x) < 0$ is feasible.
2. $\exists \lambda \geq 0$ such that $f(x) \geq \lambda g(x), \forall x \in \mathbb{R}^n$.

Recall that the Farkas lemma has a similar flavor of strong alternatives (no regularity condition is needed)

$$Ax = b, x \geq 0 \text{ is infeasible} \quad \Leftrightarrow \quad \exists y \in \mathbb{R}^m, \text{ such that } A^\mathsf{T} y \leq 0, b^\mathsf{T} y > 0.$$

There is another version of Farkas lemma that is more analogous to the S-lemma

$$a_0^\mathsf{T} x < 0, a_i^\mathsf{T} x \geq 0, i = 1, \ldots, m \text{ is infeasible} \quad \Leftrightarrow \quad \exists \lambda_i \geq 0, \text{ such that } a_0 = \sum_{i=1}^m \lambda_i a_i.$$

These are all theorems of strong alternatives that give "certificates" of infeasibility for a set of inequalities.

In later lectures, we will study the concept of sum-of-squares optimization, which can generalize the same idea to a system of polynomial equations and inequalities.

2.3 Homogeneous S-lemma

Our proof follows from [2]. We first prove the homogeneous S-lemma.

Theorem 11.2. *Suppose $\exists x_0$, such that $x_0^\top A x_0 > 0$. Then*

$$\{x \in \mathbb{R}^n \mid x^\top A x \geq 0\} \subseteq \{x \in \mathbb{R}^n \mid x^\top B x \geq 0\}$$

if and only if there exists $\lambda \geq 0$ such that $B \succeq \lambda A$.

The proof relies on the following lemma that is of interest in its own right (another key piece of the proof is the strong duality of conic programming; slater's condition holds in this case).

Lemma 11.2. *Given two symmetric matrices $P \in \mathbb{S}^n, Q \in \mathbb{S}^n$, if trace$(P) \geq 0$ and trace$(Q) < 0$, then there exists a vector $e \in \mathbb{R}^n$ such that $e^\top P e \geq 0$ and $e^\top Q e < 0$.*

Proof: For any symmetric matrix Q, we can write $Q = U^\top \Lambda U$ with an orthonormal U and a diagonal Λ. Then

$$\text{trace}(Q) = \text{trace}(U^\top \Lambda U) = \text{trace}(UU^\top \Lambda) = \text{trace}(\Lambda) < 0.$$

Now let ξ be a random vector with entries taking values ± 1 with probabilities 0.5 independently. We have

$$(U^\top \xi)^\top Q (U^\top \xi) = (U^\top \xi)^\top U^\top \Lambda U (U^\top \xi) = \xi^\top \Lambda \xi = \text{trace}(\Lambda), \quad \forall \xi.$$

On the other hand, we have

$$(U^\top \xi)^\top P (U^\top \xi) = \xi^\top U P U^\top \xi.$$

For any matrix H, we have $\mathbb{E}(\xi^\top H \xi) = \mathbb{E}(\sum_{i,j} H_{ij}\xi_i\xi_j) = \sum_{i=1}^n H_{ii}$. Therefore,

$$\mathbb{E}(\xi^\top U P U^\top \xi) = \text{trace}(UPU^\top) = \text{trace}(P) \geq 0.$$

Then, it must have $\xi \in \{-1, 1\}^n$ such that $(U^\top \xi)^\top P (U^\top \xi) \geq 0$, otherwise, the expectation would be negative. We complete the proof by taking $e = U^\top \xi$.

Proof of Theorem 11.2: We consider the following optimization problem

$$\begin{aligned} \min_x \quad & x^\top B x \\ \text{subject to} \quad & x^\top A x \geq 0 \\ & x^\top x = 1. \end{aligned} \tag{6}$$

The regularity condition implies that (6) is strictly feasible. Indeed, given $x_0 \in \mathbb{R}^n$, $x_0^\top A x_0 > 0$. We can re-scale $x = \frac{x_0}{\|x_0\|}$ which is strictly feasible to (6).

We now define $X = xx^\top$, then (6) is equivalent to

$$\begin{aligned} \min_X \quad & \langle B, X \rangle \\ \text{subject to} \quad & \langle A, X \rangle \geq 0 \\ & \text{trace}(X) = 1, X \succeq 0 \\ & \text{rank}(X) = 1. \end{aligned} \tag{7}$$

We obtain an SDP relaxation by simply dropping the rank condition, which is

$$\begin{aligned} \min_X \quad & \langle B, X \rangle \\ \text{subject to} \quad & \langle A, X \rangle \geq 0 \\ & \text{trace}(X) = 1, X \succeq 0. \end{aligned} \tag{8}$$

It dual SDP is

$$\begin{aligned} \max_{\nu,\lambda} \quad & \nu \\ \text{subject to} \quad & \lambda A + \nu I \preceq B \\ & \lambda \geq 0. \end{aligned} \tag{9}$$

It is clear that the dual SDP (9) is strictly feasible. We now argue the primal SDP (8) is also strictly feasible, by taking

$$X = \frac{x_0 x_0^\mathsf{T} + \alpha I}{\text{trace}(x_0 x_0^\mathsf{T} + \alpha I)},$$

and $\alpha > 0$ sufficiently small.

By strong duality of conic programs, we know that the optimal value of the dual SDP (9) is the same as the primal SDP (8). If we can show that the optimal value of (8) is nonnegative, we would have proven that $\exists \lambda \geq 0, \nu \geq 0$ such that $B - \lambda A \succeq \nu I \succeq 0$.

We now prove the optimal value of (8) is nonnegative using Lemma 11.2. By strong duality, we know the optimal solution X^* is attained since both primal and dual SDPs are strictly feasible. Since $X^* \succeq 0$, we write its Cholesky factorization $X^* = DD^\mathsf{T}$. We have

$$\begin{aligned} \text{trace}(AX^*) &= \text{trace}(D^\mathsf{T} A D) \geq 0 \\ \text{trace}(BX^*) &= \text{trace}(D^\mathsf{T} B D) := \theta^*. \end{aligned}$$

We need to prove $\theta^* \geq 0$. Suppose that $\theta^* < 0$. We take

$$P = D^\mathsf{T} B D, \; Q = D^\mathsf{T} A D \quad \Rightarrow \quad \text{trace}(P) \geq 0, \text{trace}(Q) < 0$$

By Lemma 11.2, there exists a vector e such that

$$\begin{aligned} e^\mathsf{T} P e \geq 0 &\Rightarrow (De)^\mathsf{T} A (De) \geq 0 \\ e^\mathsf{T} Q e < 0 &\Rightarrow (De)^\mathsf{T} B (De) < 0, \end{aligned}$$

which contradicts the hypothesis that $x^\mathsf{T} A x \geq 0 \Rightarrow x^\mathsf{T} B x \geq 0$. Therefore, we must have $\theta^* \geq 0$. This completes the proof of the homogeneous S-lemma.

2.4 Proof of the S-lemma

We now use the homogenization agreement to prove the inhomogeneous S-lemma (Theorem 11.1) based on the homogeneous S-lemma (Theorem 11.2). We define the homogeneous quadratic functions

$$\begin{aligned} \bar{f}(x,t) &= x^\mathsf{T} Q_0 x + q_0^\mathsf{T} x t + c_0 t^2 \\ \bar{g}(x,t) &= x^\mathsf{T} Q_1 x + q_1^\mathsf{T} x t + c_1 t^2 \end{aligned}$$

We argue that the condition

$$\{x \in \mathbb{R}^n \mid g(x) \geq 0\} \subseteq \{x \in \mathbb{R}^n \mid f(x) \geq 0\}$$

leads to

$$\{(x,t) \in \mathbb{R}^{n+1} \mid \bar{g}(x,t) \geq 0\} \subseteq \{(x,t) \in \mathbb{R}^{n+1} \mid \bar{f}(x,t) \geq 0\}.$$

Suppose there exist (x,t) such that $\bar{g}(x,t) \geq 0$ but $\bar{f}(x,t) < 0$.

1. If $t \neq 0$, then evaluating $(x/t, 1)$ gives a contradiction, as $g(x) = \bar{g}(x,1)$ and $f(x) = \bar{f}(x,1)$.
2. If $t = 0$ and $\bar{g}(x,t) > 0$, by continuity, there exists $t_0 \neq 0$ such that $\bar{g}(x,t_0) > 0$ and $\bar{f}(x,t_0) < 0$. Repeat Step 1.

3. If $t = 0$ and $\bar{g}(x,t) = 0$. In this case, we have $x^\mathsf{T} Q_1 x = 0$ and $x^\mathsf{T} Q_0 x < 0$. Now, we slightly change $t_0 \neq 0$ such that $\bar{f}(x,t_0) < 0$. We then rescale x to γx, such that
   - In $\bar{g}(\gamma x, t_0)$, the quadratic term $\gamma^2 x^\mathsf{T} Q_1 x = 0$ and the linear term $q_1^\mathsf{T}(\gamma x) t_0$ becomes positive and dominates the constant term.
   - In $\bar{f}(\gamma x, t_0)$, the quadratic term $\gamma^2 x^\mathsf{T} Q_0 x$ is negative and dominate the other terms.

   Thus, we find $(\gamma x, t_0)$, such tat $\bar{g}(\gamma x, t_0) \geq 0$ but $\bar{f}(\gamma x, t_0) < 0$ with $t_0 \neq 0$. Repeat Step 1.

Also, it is clear the regularity condition for the homogeneous version still holds. We can now apply the homogeneous S-lemma, which tells that

$$\exists \lambda \geq 0, \quad \text{such that} \quad \bar{f}(x,t) \geq \lambda \bar{g}(x,t),\ \forall x, t.$$

Setting $t = 1$, we get the desired result:

$$\exists \lambda \geq 0, \quad \text{such that} \quad f(x) \geq \lambda g(x),\ \forall x.$$

3 Shor's semidefinite relaxation and the Lagrange relaxation

Consider the non-convex QCQP

$$\begin{aligned} \min_{x} \quad & x^\mathsf{T} Q_0 x + q_0^\mathsf{T} x + c_0 \\ \text{subject to} \quad & x^\mathsf{T} Q_i x + q_i^\mathsf{T} x + c_i \leq 0, i = 1, \ldots, m \end{aligned} \tag{10}$$

where $Q_0, Q_i \in \mathbb{S}^n, q_0, q_i \in \mathbb{R}^n, c_0, c_i \in \mathbb{R}$. We introduce a matrix variable $X = xx^\mathsf{T}$. The problem (10) becomes

$$\begin{aligned} \min_{x} \quad & \langle Q_0, X \rangle + q_0^\mathsf{T} x + c_0 \\ \text{subject to} \quad & \langle Q_i, X \rangle + q_i^\mathsf{T} x + c_i \leq 0, i = 1, \ldots, m \\ & X = xx^\mathsf{T}. \end{aligned} \tag{11}$$

We now relax the constraint $X = xx^\mathsf{T}$ to a convex constraint $X \succeq xx^\mathsf{T}$, which is equivalent to (by Schur complement)

$$\begin{bmatrix} 1 & x^\mathsf{T} \\ x & X \end{bmatrix} \succeq 0.$$

We then obtain a semidefinite relaxation of (10)

$$\begin{aligned} \min_{x} \quad & \langle Q_0, X \rangle + q_0^\mathsf{T} x + c_0 \\ \text{subject to} \quad & \langle Q_i, X \rangle + q_i^\mathsf{T} x + c_i \leq 0, i = 1, \ldots, m \\ & \begin{bmatrix} 1 & x^\mathsf{T} \\ x & X \end{bmatrix} \succeq 0. \end{aligned} \tag{12}$$

Another way to derive a convex relaxation is based on the Lagrange dual formulation. The Lagrangian of (10) is

$$\begin{aligned} L(x,\lambda) &= x^\mathsf{T} Q_0 x + q_0^\mathsf{T} x + c_0 + \sum_{i=1}^m \lambda_i (x^\mathsf{T} Q_i x + q_i^\mathsf{T} x + c_i) \\ &= x^\mathsf{T} \left(Q_0 + \sum_{i=1}^m \lambda_i Q_i \right) x + \left(q_0 + \sum_{i=1}^m \lambda_i q_i \right)^\mathsf{T} x + c_0 + \sum_{i=1}^m \lambda_i c_i. \end{aligned}$$

The dual function is

$$g(\lambda) = \min_x L(x, \lambda).$$

If

$$L(x, \lambda) - \xi \geq 0, \forall x \in \mathbb{R}^n, \tag{13}$$

then $g(\lambda) \geq \xi$. The condition (13) is equivalent to

$$\begin{bmatrix} c_0 + \sum_{i=1}^m \lambda_i c_i - \xi & \frac{1}{2}\left(q_0 + \sum_{i=1}^m \lambda_i q_i\right)^\mathsf{T} \\ \frac{1}{2}\left(q_0 + \sum_{i=1}^m \lambda_i q_i\right) & Q_0 + \sum_{i=1}^m \lambda_i Q_i \end{bmatrix} \succeq 0.$$

Therefore, we get a dual problem for (10) as follows

$$\begin{aligned} \max_{\xi, \lambda} \quad & \xi \\ \text{subject to} \quad & \begin{bmatrix} c_0 + \sum_{i=1}^m \lambda_i c_i - \xi & \frac{1}{2}\left(q_0 + \sum_{i=1}^m \lambda_i q_i\right)^\mathsf{T} \\ \frac{1}{2}\left(q_0 + \sum_{i=1}^m \lambda_i q_i\right) & Q_0 + \sum_{i=1}^m \lambda_i Q_i \end{bmatrix} \succeq 0, \\ & \lambda \geq 0. \end{aligned} \tag{14}$$

One can verify that (14) is just the semidefinite dual of (12). We call both (12) and (14) semidefinite relaxations of the non-convex QCQP (10).

Notes

The preparation of this lecture is based on [1, Lecture 12]. Further reading for this lecture can refer to [2, Chapter 4.3] and [3, Appendix B].

References

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[2] Aharon Ben-Tal and Arkadi Nemirovski. *Lectures on modern convex optimization: analysis, algorithms, and engineering applications.* SIAM, 2001.

[3] Stephen Boyd, Stephen P Boyd, and Lieven Vandenberghe. *Convex optimization.* Cambridge university press, 2004.

[4] Imre Pólik and Tamás Terlaky. A survey of the s-lemma. *SIAM review*, 49(3):371–418, 2007.