ECE285: Semidefinite and sum-of-squares optimization **Winter 2024**

Lecture 13: Nonnegative polynomials, SOS, and SDPs (I)

Lecturer: Yang Zheng *Scribe: Yang Zheng*

Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications. Any typos should be sent to* `zhengy@eng.ucsd.edu`.

Learning goals:

1. Nonnegative univariate polynomials
2. Sum-of-squares polynomials
3. Nonnegativity on intervals

From this lecture, we begin our study of another main theme of the course, i.e., nonnegative polynomials, and their relationship with sums of squares and semidefinite programming. In this lecture, we mainly focus on univariate polynomials $p(x) : \mathbb{R} \to \mathbb{R}$. We denote the set of univariate polynomials as $\mathbb{R}[x]$.

1 Nonnegative polynomials

We say a polynomial $p \in \mathbb{R}[x]$ is nonnegative if $p(x) \geq 0, \forall x \in \mathbb{R}$. Checking the nonnegativity of polynomials has many applications. For instance, if we have a "good" way to check the nonnegativity of polynomials, then we can also minimize or maximize polynomials:

$$\min_{x\in\mathbb{R}} \; p(x) \quad \Longleftrightarrow \quad \begin{array}{rl} \max & \gamma \\ \text{subject to} & p(x) - \gamma \geq 0, \forall x \in \mathbb{R}. \end{array} \tag{1}$$

Given any nonnegative univariate polynomial $p(x)$, we can verify that

- Its degree $p(x)$ is even and the leading coefficient (i.e., the coefficient of x^{2d}) is nonnegative;
- Any real root of $p(x)$ has even multiplicity.

Indeed, these two conditions are also sufficient. We now introduce a very useful notion of sum-of-squares.

Definition 13.1. *A univariate polynomial $p(x) = \sum_{k=0}^{2d} p_k x^k$ of degree $2d$ is a sum of squares (SOS) if there exist polynomials $q_1(x), \ldots, q_t(x)$ such that*

$$p(x) = \sum_{k=1}^{t} q_k^2(x). \tag{2}$$

It is clear that a sum-of-squares polynomial $p(x)$ is nonnegative globally. For univariate polynomials, the converse is also true.

Theorem 13.1. *A univariate polynomial $p(x) = \sum_{k=0}^{2d} p_k x^k$ of degree $2d$ is nonnegative if and only if there exist $q_1(x)$ and $q_2(x)$ of degree no greater than d such that*

$$p(x) = q_1^2(x) + q_2^2(x).$$

Proof. The "if" part is obvious. We prove the "only if" part. Assume $p(x)$ is nonnegative. Since $p(x)$ has real coefficients, if $p(z) = 0$, then we must have $p(\bar{z}) = 0$. Also, if z is a real root of $p(x)$, it must have even multiplicity. These two facts, together with the fundamental theorem of algebra, lead to

$$p(x) = p_{2d} \prod_{i=1}^{d} (x - z_i)(x - \bar{z}_i) = \left| \sqrt{p_{2d}} \prod_{i=1}^{d} (x - z_i) \right|^2 .$$

Upon defining

$$q(x) = \sqrt{p_{2d}} \prod_{i=1}^{d} (x - z_i),\ q_1(x) = \mathrm{Re}[q(x)],\ q_2(x) = \mathrm{Im}[q(x)],$$

we get the desired result: $p(x) = q_1^2(x) + q_2^2(x)$. □

2 SOS and semidefinite programming

Consider a univariate polynomial $p(x) = p_0 + p_1 x + p_2 x^2 + \ldots + p_{2d} x^{2d}$ that is SOS, i.e., it can be written in the form of (2). Note that the degree of $q_k(x)$ is at most d since the highest term of each $q_k^2(x)$ is positive and there cannot be any cancellation in the highest power of x. Therefore, we can write

$$\begin{bmatrix} q_1(x) \\ q_2(x) \\ \vdots \\ q_t(x) \end{bmatrix} = V \begin{bmatrix} 1 \\ x \\ \vdots \\ x^d \end{bmatrix},$$

where $V \in \mathbb{R}^{t \times (d+1)}$ and its kth row contains the coefficients of the polynomial $q_k(x)$. For convenience, we denote

$$[x]_d := \left[1, x, \ldots, x^d\right]^{\mathsf{T}}.$$

We can then rewrite (2) into

$$p(x) = \begin{bmatrix} q_1(x) \\ q_2(x) \\ \vdots \\ q_t(x) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} q_1(x) \\ q_2(x) \\ \vdots \\ q_t(x) \end{bmatrix} = (V[x]_d)^{\mathsf{T}} V[x]_d = [x]_d^{\mathsf{T}} V^{\mathsf{T}} V [x]_d = [x]_d^{\mathsf{T}} Q [x]_d, \tag{3}$$

where $Q = V^{\mathsf{T}} V$ is positive semidefinite. Conversely, assume $Q \succeq 0$ satisfying (3). Then we can factorize $Q = V^{\mathsf{T}} V$, and we arrive at an SOS representation of $p(x)$.

The following theorem gives a direct relationship between positive semidefinite matrices and nonnegative univariate polynomials.

Theorem 13.2. *Let $p(x) = \sum_{k=0}^{2d} p_k x^k$ be a univariate polynomial. Then, $p(x)$ is nonnegative (or equivalently SOS) if and only if there exists $Q \in \mathbb{S}_+^{d+1}$ that satisfies*

$$p(x) = [x]_d^{\mathsf{T}} Q [x]_d. \tag{4}$$

If we index the rows and columns of the matrix Q by $0, \ldots, d$, we have

$$[x]_d^{\mathsf{T}} Q [x]_d = \sum_{i=0}^{d} \sum_{j=0}^{d} Q_{ij} x^{i+j} = \sum_{k=0}^{2d} \left(\sum_{i+j=k} Q_{ij} \right) x^k.$$

Matching the coefficients of the left- and right- hand sides in (4), we have

$$p_k = \sum_{i+j=k} Q_{ij}, \qquad k = 0, \dots, 2d.$$

This is a system of $2d+1$ linear equations. Therefore, the matrix Q is constrained to be positive semidefinite and be in a particular affine space. An SOS condition is equivalent to a semidefinite programming problem.

Example 13.1. *[1, Example 3.35] Consider the univariate polynomial*

$$p(x) = x^4 + 4x^3 + 6x^2 + 4x + 5.$$

We aim to find an SOS decomposition. Consider the expression

$$\begin{aligned} p(x) &= \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^\mathsf{T} \underbrace{\begin{bmatrix} q_{00} & q_{01} & q_{02} \\ q_{10} & q_{11} & q_{12} \\ q_{20} & q_{21} & q_{22} \end{bmatrix}}_{Q} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \\ &= q_{22}x^4 + 2q_{12}x^3 + (q_{11} + 2q_{02})x^2 + 2q_{01}x + q_{00}. \end{aligned}$$

Matching the coefficients, we have the following linear equations

$$\begin{aligned} x^4 &: \ 1 = q_{22}, \\ x^3 &: \ 4 = 2q_{12}, \\ x^2 &: \ 6 = q_{11} + 2q_{02}, \\ x &: \ 4 = 2q_{01}, \\ 1 &: \ 5 = q_{00}. \end{aligned}$$

We need to find a positive semidefinite matrix Q satisfying the equations above (i.e. solving an SDP). In this case, a feasible solution is given by

$$Q = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 1 \end{bmatrix} = V^\mathsf{T} V, \qquad V = \begin{bmatrix} 0 & 2 & 1 \\ \sqrt{2} & \sqrt{2} & 0 \\ \sqrt{3} & 0 & 0 \end{bmatrix}.$$

This gives the following SOS decomposition: $p(x) = (x^2 + 2x)^2 + 2(x+1)^2 + 3$. □

Theorem 13.2 allows us to check the nonnegativity of univariate polynomials using semidefinite programming. Indeed, we can minimize any univariate polynomial (even if it is nonconvex) using semidefinite programming. It is clear that (1) is equivalent to

$$\begin{aligned} \max_{\gamma, Q} \quad & \gamma \\ \text{subject to} \quad & p_0 - \gamma = Q_{00}, \\ & p_k = \sum_{i+j=k} Q_{ij}, \ k = 1, \dots, 2d, \\ & Q \in \mathbb{S}_+^{d+1}. \end{aligned} \tag{5}$$

Example 13.2. *It is well-known that a second-order polynomial $p(x) = ax^2 + bx + c$ is nonnegative globally if and only if its discriminant $\Delta := b^2 - 4ac \leq 0$ and $a, c \geq 0$. Theorem 13.2 ensures that this polynomial $p(x)$ is nonnegative globally if and only if there exists a matrix $Q \in \mathbb{S}_+^2$ such that*

$$Q_{00} = c, \ \ Q_{01} + Q_{10} = b, \ \ Q_{11} = a.$$

This is equivalent to

$$\begin{bmatrix} c & \frac{b}{2} \\ \frac{b}{2} & a \end{bmatrix} \succeq 0 \quad \Longleftrightarrow \quad b^2 - 4ac \leq 0, \quad a, c \geq 0.$$

We define the cone of nonnegative univariate polynomials of degree 2d as

$$P_{2d} = \left\{ (p_0, p_1, \ldots, p_{2d}) \in \mathbb{R}^{2d+1} \mid \sum_{k=0}^{2d} p_k x^k \geq 0, \forall x \in \mathbb{R} \right\}.$$

It is easy to see that

$$P_{2d} = \bigcap_{x \in \mathbb{R}} \underbrace{\left\{ (p_0, p_1, \ldots, p_{2d}) \in \mathbb{R}^{2d+1} \mid \sum_{k=0}^{2d} p_k x^k \geq 0 \right\}}_{H_x},$$

where each H_x is a closed halfspace in $\mathbb{R}^{2d+1}$. P_{2d} is closed and convex since it is an (infinitely) intersection of closed convex sets. If $p(x) \geq 0, \forall x \in \mathbb{R}$ and $-p(x) \geq 0, \forall x \in \mathbb{R}$, it must be identically zero, $p(x) \equiv 0$. Thus, P_{2d} is pointed. One can check that $x^{2d} + 1$ is an interior point of P_{2d}. In summary, we have the following result.

Lemma 13.1. *P_{2d} is a proper cone in $\mathbb{R}^{2d+1}$.*

Similarly, one can further define a conic program over P_{2d}:

$$\begin{aligned} \min_{p} \quad & c^\intercal p \\ \text{subject to} \quad & Ap = b, \\ & p \in P_{2d}. \end{aligned}$$

This conic program can be equivalently reformulated into an SDP. For instance, (1) can be written as

$$\begin{aligned} \min_{\gamma} \quad & \gamma \\ \text{subject to} \quad & (p_0 - \gamma, p_1, \ldots, p_{2d}) \in P_{2d}. \end{aligned}$$

Remark 13.1. *Later in this course, we will discuss the dual of P_{2d}, which has an interesting interpretation as moment problems.* □

3 Nonnegativity over intervals

We have characterized the univariate polynomials that are nonnegative globally on the real line. Here, we focus on polynomials nonnegative on an interval. In particular, we have the following necessary and sufficient conditions for a univariate polynomial to be nonnegative on $[-1, 1]$.

Theorem 13.3. *Consider a univariate polynomial $p(x)$.*

- *If $p(x)$ is of even degree $2d$, then $p(x) \geq 0, \forall x \in [-1, 1]$ if and only if there exists SOS polynomial $s_1(x)$ of degree $2d$ and $s_2(x)$ of degree $2d - 2$ such that*

$$p(x) = s_1(x) + (1 - x^2)s_2(x) \tag{6}$$

- *If $p(x)$ is of odd degree $2d + 1$, then $p(x) \geq 0, \forall x \in [-1, 1]$ if and only if there exists SOS polynomial $s_1(x)$ of degree $2d$ and $s_2(x)$ of degree $2d$ such that*

$$p(x) = (1 - x)s_1(x) + (1 + x)s_2(x) \tag{7}$$

Note that the "if" part is obvious, and the "only if" part is slightly more difficult but it can be proved by induction. We leave it as an exercise. The condition (6) can be seen as an algebraic proof of the nonnegativity of $p(x)$ on $[-1, 1]$.

Remark 13.2. *The case with nonnegativity on $[a, b]$ can be reduced to Theorem 13.3 by a change of variables*

$$f(x) := p\left(\frac{2x - (a+b)}{b-a}\right).$$

Indeed, we can establish similar necessary and sufficient conditions: $p(x)$ is nonnegative on $[a, b]$ if and only if

$$\begin{cases} p(x) = s_1(x) + (x-a)(b-x)s_2(x), & \text{if } deg(p) \text{ is even} \\ p(x) = (x-a)s_1(x) + (b-x)s_2(x), & \text{if } deg(p) \text{ is odd} \end{cases}$$

where $s_1(x)$, $s_2(x)$ are SOS. If $deg(p) = 2d$, we have $deg(s_1) \leq 2d$ and $deg(s_2) \leq 2d-2$; if $deg(p) = 2d+1$, we have $deg(s_1) \leq 2d$ and $deg(s_2) \leq 2d$. □

It is important to note that both (6) and (7) can be searched using semidefinite programming, since $p(x) = s_1(x) + (1-x^2)s_2(x)$ defines a set of linear constraints on the coefficients of $s_1(x)$ and $s_2(x)$ (similarly for $p(x) = (1-x)s_1(x) + (1+x)s_2(x)$).

Let $P_{2d}[-1, 1]$ be the cone of polynomials of degree $2d$ nonnegative on $[-1, 1]$. Then, we can represent this set as

$$P_{2d}[-1,1] = \{(p_0, \ldots, p_{2d}) \in \mathbb{R}^{2d+1} \mid \exists s_1 \in P_{2d}, s_2 \in P_{2d-2}, \text{ s.t. } p(x) = s_1(x) + (1-x^2)s_2(x)\}.$$

Testing $p \in P_{2d}[-1, 1]$ is a semidfinite program. Furthermore, the following problem

$$\begin{aligned} \min_x \quad & p(x) \\ \text{subject to} \quad & -1 \leq x \leq 1 \end{aligned}$$

can be expressed as a semidefinite program in the following way

$$\begin{aligned} \max_\gamma \quad & \gamma \\ \text{subject to} \quad & p(x) - \gamma \geq 0, \forall x \in [-1,1] \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \max_\gamma \quad & \gamma \\ \text{subject to} \quad & (p_0 - \gamma, p_1, \ldots, p_{2d}) \in P_{2d}[-1,1]. \end{aligned}$$

Theorem 13.4. *Consider a univariate polynomial $p(x)$. We have $p(x) \geq 0, \forall x \in [0, \infty)$ if and only if there exist SOS polynomials $s_1(x), s_2(x)$ such that*

$$p(x) = s_1(x) + x s_2(x),$$

with degree bounds as

- *$deg(s_1) \leq 2d$ and $deg(s_2) \leq 2d-2$ if $deg(p) = 2d$ (even);*
- *$deg(s_1) \leq 2d$ and $deg(s_2) \leq 2d$ if $deg(p) = 2d+1$ (odd);*

Example 13.3. *Consider the following SOS program*

$$\begin{aligned} \max_\gamma \quad & \gamma \\ \textit{subject to} \quad & p(x) - \gamma \quad \textit{is SOS}. \end{aligned}$$

Suppose $p(x) = x^4 + 2x^3 - 3x^2 - 4x + 5$. YALMIP code is

```
%  SOS example 1
x     = sdpvar(1,1);                    % define variable
gamma = sdpvar(1,1);
p     = x^4 +2*x^3 - 3*x^2 - 4*x + 5;   % polynomial
F     = sos(p-gamma);                   % SOS constraint
solvesos(F,-gamma,[],gamma);            % solve SOS program
value(gamma)
```

Suppose $p(x) = x_1^4 + 2x_2^4 - 2x_1^2x_2^2 - 2x_1x_2^2 + x_1^2 + 4$*. YALMIP code is*

```
%  SOS example 2
x     = sdpvar(1,1);                    % define variable
gamma = sdpvar(1,1);
p = x(1)^4 +2*x(2)^4 - 2*x(1)^2*x(2)^2 - 2*x(1)*x(2)^2 + x(1)^2 +4;   % polynomial
F     = sos(p-gamma);                   % SOS constraint
solvesos(F,-gamma,[],gamma);            % solve SOS program
value(gamma)
```

Notes

The preparation of this lecture was based on [2, Lectures 10 & 12]. Further reading for this lecture can refer to [1, Chapter 3].

References

[1] Grigoriy Blekherman, Pablo A Parrilo, and Rekha R Thomas. *Semidefinite optimization and convex algebraic geometry*. SIAM, 2012.

[2] Hamza Fawzi. Topics in Convex Optimisation, Michaelmas 2018.