ECE285: Semidefinite and sum-of-squares optimization

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Lecture 14: Nonnegative polynomials, SOS, and SDPs (II)

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#### Learning goals:

- 1. Nonnegative multivariate polynomials
- 2. Sum-of-squares hierarchies
- 3. Nonnegativity on Sets: Positivstellensatz

# **1** Nonnegative multivariate polynomials

In this lecture, we start to look at polynomials in more than one variable. We denote by  $\mathbb{R}[x]$  the space of polynomials in n variables  $x_1, \ldots, x_n$ . A monomial is expressed as  $x^{\alpha} := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$  with  $x = (x_1, \ldots, x_n)$  and  $\alpha = (\alpha_1, \ldots, \alpha_n)$ , where  $\alpha_i, i = 1, \ldots, n$  are integers. The degree of a monomial  $x^{\alpha}$  is  $|\alpha| := \alpha_1 + \alpha_2 + \ldots + \alpha_n$ . The degree of a polynomial is the largest degree of its monomials. For example,

$$p(x) = x_1^3 + 2x_1x_2^2 + x_1x_2 + 2 \tag{1}$$

has degree 3. We denote by  $\mathbb{R}[x]_{n,d}$  the space of polynomials in *n* variables of degree no bigger than *d*. The canonical monomial basis for  $\mathbb{R}[x]_{n,d}$  is

$$[x]_{d} = \begin{bmatrix} 1, x_{1}, x_{2}, \dots, x_{n}, x_{1}^{2}, x_{1}x_{2}, \dots, x_{n}^{2}, \dots, x_{1}^{d}, \dots, x_{n}^{d} \end{bmatrix}^{\mathsf{T}},$$

which has size  $\binom{n+d}{d}$ . Each element  $p(x) = \sum_{|\alpha| \leq d} p_{\alpha} x^{\alpha} \in \mathbb{R}[x]_{n,d}$  can be identified by its coefficients  $p_{\alpha}, |\alpha| \leq d$ . For example, when n = 2, d = 3, we have

$$[x]_d = \begin{bmatrix} 1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3 \end{bmatrix}^{\mathsf{T}},$$

and the polynomial in (1) reads as

$$p(x) = [2, 0, 0, 0, 1, 0, 1, 0, 2, 0] [x]_d.$$

We are interested in polynomials p(x) that are nonnegative globally on  $\mathbb{R}^n$ , i.e.,  $p(x) \ge 0, \forall x \in \mathbb{R}^n$ . An obvious sufficient condition for nonnegative polynomials is a sum-of-squares (SOS) representation.

**Definition 14.1.** A polynomial  $p(x) \in \mathbb{R}[x]$  is a sum of squares if there exists polynomials  $q_1(x), \ldots, q_t(x) \in \mathbb{R}[x]$  such that

$$p(x) = q_1(x)^2 + \ldots + q_t(x)^2.$$

It is not difficult to see that if  $p(x) \in \mathbb{R}[x]$  is nonnegative globally, it must have ven degree. If p(x) is of even degree 2d and  $p(x) = \sum_{k=1}^{t} q_k(x)^2$ , then we always have  $\deg(q_k) \leq d, k = 1, \ldots, t$ .

In Lecture 11, we have seen that for univariate polynomials (n = 1), nonnegativity is equivalent to an SOS condition. It turns out that in general this is not true. In fact, for polynomials of degree no less than four, deciding whether a polynomial is nonnegative globally is NP-hard (as a function of the number of variables).

Let  $P_{n,2d}$  be the cone of nonnegative polynomials in n variables of degree at most 2d. Let  $\Sigma_{n,2d}$  be the cone of SOS polynomials in n variables of degree at most 2d. More than a century ago, David Hilbert showed the equivalence between nonnegative polynomials and SOS polynomials (i.e.,  $\Sigma_{n,2d} = P_{n,2d}$ ) holds only in the following three cases.

**Theorem 14.1** (Hilbert).  $\Sigma_{n,2d} = P_{n,2d}$  holds only in the following three cases:

- n = 1 (univariate polynomials);
- 2d = 2 (quadratic polynomials);
- n = 2, 2d = 4 (bivariate quartics).

We have already seen that  $\Sigma_{n,2d} = P_{n,2d}$  in the case of n = 1. The case 2d = 2 can be proved easily using the eigenvalue decomposition of positive semidefinite matrices. The last case n = 2, 2d = 4 is more difficult.

Checking  $p(x) \in P_{n,2d}$  is computationally hard. On the other hand, checking  $p(x) \in \Sigma_{n,2d}$  can be performed using semidefinite programs. The following result is a multivariate generalization of the result in the previous lecture. For convenience, we write  $s(n,d) = \binom{n+d}{d}$ .

Theorem 14.2. Consider a polynomial of degree 2d in the form of

$$p(x) = \sum_{|\alpha| \le 2d} p_{\alpha} x^{\alpha}.$$

Then  $p(x) \in \Sigma_{n,2d}$  if and only if there exists a positive semidefinite matrix  $Q \in \mathbb{S}^{s(n,d)}_+$  such that

$$p_{\alpha} = \sum_{\beta + \gamma = \alpha} Q_{\beta\gamma}, \quad \forall |\alpha| \le 2d.$$
<sup>(2)</sup>

*Proof.* We first prove "Only if" part. Suppose  $p(x) \in \Sigma_{n,2d}$ . There exist  $q_1(x), \ldots, q_t(x) \in \mathbb{R}[x]_{n,d}$  such that

$$p(x) = \begin{bmatrix} q_1(x) \\ \vdots \\ q_t(x) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} q_1(x) \\ \vdots \\ q_t(x) \end{bmatrix}.$$
(3)

Upon choosing the monomial basis  $[x]_d$  and collecting all the coefficients of  $q_1(x), \ldots, q_t(x)$ , we have

$$\begin{bmatrix} q_1(x) \\ \vdots \\ q_t(x) \end{bmatrix} = V[x]_d.$$

Then, (3) becomes

$$p(x) = (V[x]_d)^{\mathsf{T}} V[x]_d = [x]_d (V^{\mathsf{T}} V) [x]_d.$$

Therefore, we obtain a positive semidefinite matrix  $Q = V^{\mathsf{T}} V \in \mathbb{S}^{s(n,d)}_+$ . The linear equations are obtained by matching coefficients

$$p(x) = [x]_d Q[x]_d = \sum_{|\alpha| \le 2d} \left( \sum_{\beta + \gamma = \alpha} Q_{\beta\gamma} \right) x^{\alpha}.$$

"If" part is clear by reversing the arguments above based on a Cholesky factorization of  $Q = V^{\mathsf{T}}V$ .

**Remark 14.1.** Similar to the univariate case, deciding the membership  $p(x) \in \Sigma_{n,2d}$  or performing linear optimization over  $\Sigma_{n,2d}$  is equivalent to solving an SDP.

**Example 14.1** ([1, Example 3.38]). Consider a multivariate polynomial

$$p(x,y) = 2x^4 + 5y^4 - x^2y^2 + 2x^3y + 2x + 2.$$

We want to find an SOS representation for this polynomial. In this case, n = 2 and 2d = 4, the monomial basis is

$$[x]_d = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 \end{bmatrix}^{\mathsf{T}}$$

and the matrix Q, indexed by this monomial basis, is

$$Q = \begin{bmatrix} q_{00,00} & q_{00,10} & q_{00,01} & q_{00,20} & q_{00,11} & q_{00,02} \\ q_{10,10} & q_{10,01} & q_{10,20} & q_{10,11} & q_{10,02} \\ q_{01,01} & q_{01,20} & q_{01,11} & q_{01,02} \\ q_{20,20} & q_{20,11} & q_{20,02} \\ q_{11,11} & q_{11,02} \\ q_{02,02} \end{bmatrix}$$
(4)

(The rest of entries are symmetrical.) Checking whether p(x, y) is SOS is equivalent to deciding whether there is a matrix Q in (4) that is positive semidefinite and satisfies the linear constraints (2). In this case, there is a total of  $s(n, 2d) = {6 \choose 4} = 15$  linear constraints, one for each monomial  $x^{\alpha}$  of degree at most 2d. For instance, the equations corresponding to monomials  $x^4$ ,  $x^2y^2$ , and  $y^2$  are

$$x^{4}: \qquad 2 = q_{20,20}$$
  

$$x^{2}y^{2}: \qquad -1 = 2q_{20,02} + q_{11,11}$$
  

$$y^{2}: \qquad 0 = 2q_{00,02} + q_{01,01}$$

This amounts to solving a feasibility of an SDP. One feasible solution is given as

$$Q = \frac{1}{3} \begin{bmatrix} 6 & 3 & 0 & -2 & 0 & -2 \\ 4 & 0 & 0 & 0 & 0 \\ & 4 & 0 & 0 & 0 \\ & & 6 & 3 & -4 \\ & & & 5 & 0 \\ & & & & 15 \end{bmatrix}$$

Any factorization of matrix Q will lead to an SOS decomposition. For instance, one SOS decomposition is

$$p(x,y) = \frac{4}{3}y^2 + \frac{1349}{705}y^4 + \frac{1}{12}(4x+3)^2 + \frac{1}{15}(3x^2+5xy)^2 + \frac{1}{315}(-21x^2+20y^2+10)^2 + \frac{1}{59220}(328y^2-235)^2.$$

Theorem 14.1 has identified all the cases where nonnegativity is equivalent to an SOS condition. For all other cases, there always exist nonnegative polynomials that are not sum-of-squares. Consider the Motzkin polynomial (n = 2, 2d = 6)

$$M(x,y) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2.$$

One can show that M(x,y) is nonnegative globally via the arithmetic-geometric mean inequality:

$$\frac{1}{3}(x^4y^2 + x^2y^4 + 1) \ge (x^6y^6)^{1/3} = x^2y^2, \qquad \forall x, y \in \mathbb{R}$$

On the other hand, one can show that M(x, y) is not a sum of squares. In fact, one can prove that  $M(x, y) - \gamma$  is not a sum of squares for any  $\gamma \in \mathbb{R}$ .

**Proposition 14.1.**  $M(x,y) - \gamma$  is not a sum of squares for any  $\gamma \in \mathbb{R}$ .

### 2 Sum-of-squares hierarchies

Consider a global polynomial optimization problem

$$p^* = \min_{x \in \mathbb{R}^n} \quad p(x) \tag{5}$$

where  $p(x) \in \mathbb{R}[x]_{n,2d}$ . As we have seen last time, this is equivalent to

$$\begin{split} \gamma^{\star} &= \max_{\gamma} \quad \gamma \\ \text{subject to} \quad p(x) - \gamma \geq 0, \qquad \forall x \in \mathbb{R}^{n}. \end{split}$$

We can establish a lower bound by solving an SDP as

$$\begin{array}{ll} \gamma_0 = \max_{\gamma} & \gamma \\ \text{subject to} & p(x) - \gamma \in \Sigma_{n,2d}. \end{array}$$

We have  $\gamma_0 \leq \gamma^* = p^*$ . In the case of univariate polynomials (n = 1), we have  $\gamma_0 = \gamma^*$  since nonnegative univariate polynomials are SOS. However, we have a strict inequality in general. Recall the Motzkin polynomial  $M(x, y) - \gamma$  is not SOS for any  $\gamma$ . In this case,  $\gamma_0 = -\infty$ . This lower bound from the semidefinite relaxation is not useful. On the other hand, we can verify that

$$(1+x^2+y^2)M(x,y) = y^2(1-x^2)^2 + x^2(1-y^2)^2 + (x^2y^2-1)^2 + x^2y^2\left(\frac{3}{4}(x^2+y^2-2)^2 + \frac{1}{4}(x^2-y^2)^2\right).$$

This also clearly shows that  $M(x, y) \ge 0, \forall (x, y) \in \mathbb{R}^2$ .

The fact above actually suggests a hierarchy of sum-of-squares relaxations for (5)

$$\gamma_r = \max_{\gamma} \quad \gamma$$
subject to  $(1 + x_1^2 + \ldots + x_n^2)^r (p(x) - \gamma) \in \Sigma_{n,2d+2r}.$ 
(6)

It is not hard to show that  $\gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \ldots \leq p^*$ . Note that one can define another hierarchy of semidefinite relaxations where the multiplier  $(1 + x_1^2 + \ldots + x_n^2)^r$  is replaced by another nonnegative polynomial. This will lead to a different hierarchy.

One natural question is to ask whether the sequence  $\gamma_r$  defined in (6) converges to  $p^*$ . Under some assumptions on p(x), one can actually establish the convergence result. For example, we have the following result, established by Reznick [4] (a homogeneous polynomial of degree 2d is a polynomial with only monomials of degree exactly 2d, e.g.,  $p(x_1, x_2) = x_1^4 + x_2^4 + x_1^2 x_2^2$ ).

**Theorem 14.3** (Reznick's theorem). Let  $p(x) \in \mathbb{R}[x]_{n,2d}$  be a homogeneous polynomial. If  $p(x) > 0, \forall x \in \mathbb{R}^n \setminus \{0\}$ , then there exists  $r \in \mathbb{N}$  such that

$$(x_1^2 + \dots x_n^2)^r p(x)$$

is a sum of squares.

Note that  $(x_1^2 + \ldots x_n^2)^r p(x)$  being SOS means that p(x) can be written as a sum of squares of rational functions. Hilbert's 17th problem asks whether any nonnegative polynomial can be written a sum of squares of rational functions. This question was answered positively by Artin in 1927.

**Theorem 14.4** (Hilbert–Artin's theorem). Let  $p(x) \in \mathbb{R}[x]_{n,2d}$ . If  $p(x) \ge 0, \forall x \in \mathbb{R}^n$ , then there exist nonzero SOS polynomials h(x) and q(x) such that

$$h(x)p(x) = q(x).$$

## 3 Nonnegativity on Sets

An SOS representation is an obvious certificate of the global nonnegativity of a polynomial p(x) over the entire space  $\mathbb{R}^n$ . In this section, we look into the nonnegativity of a polynomial p(x) on a given subset  $S \in \mathbb{R}^n$ , i.e., we aim to provide certificates for

$$p(x) \ge 0, \quad \forall x \in S. \tag{7}$$

The set S could be defined in different forms, and the certificates we discuss below will depend on how S is presented.

#### 3.1 Equations

Consider the set S is defined by a set of polynomial equations

$$S = \{ x \in \mathbb{R}^n \mid f_1(x) = 0, \dots, f_m(x) = 0 \}.$$

Recalling the form of Lagrange multipliers, we can write the following condition

$$p(x) + \sum_{i=1}^{m} \lambda_i(x) g_i(x) \quad \text{is SOS,}$$
(8)

where  $\lambda_i(x)$  are arbitrary polynomials. If we can find  $\lambda_i(x)$  such that (8) holds, then it is obviously a certificate to establish (7). Indeed, if (8), by evaluating this expression at any point  $x_0 \in S$ , we conclude that  $p(x_0) \geq 0$ . Note that (8) is affine in the unknown polynomials  $\lambda_i(x)$ . Once we fix the degree of  $\lambda_i(x)$ , (8) is amount to solving an SDP.

#### 3.2 Inequalities

Consider the set S is defined by a set of polynomial inequalities

$$S = \{ x \in \mathbb{R}^n \mid g_1(x) \ge 0, \dots, g_m(x) \ge 0 \}.$$

We have very similar arguments. Inspired by Lagrange multipliers, we consider

$$p(x) = s_0(x) + \sum_{i=1}^m s_i(x)g_i(x),$$
(9)

where  $s_0(x)$  and  $s_i(x)$  are SOS polynomials. This condition (9) serves a obvious certificate for (7). Indeed, evaluating the expression (9) at any point  $x_0 \in S$  directly proves  $p(x_0) \ge 0$ . Again, (9) is affine in the unknown polynomials  $s_0(x), s_i(x)$ . Once we fix the degree of  $s_0(x), s_i(x)$ , (9) is amount to solving an SDP.

**Remark 14.2.** We can also consider more powerful expressions by allowing finite products in the following form

$$p(x) = s_0(x) + \sum_{i=1}^m s_i(x)g_i(x) + \sum_{i,j}^m s_{ij}(x)g_i(x)g_j(x) + \dots$$

where  $s_0(x), s_i(x), s_{ij}(x)$  are SOS polynomials. This representation is also an obvious certificate for (7). Upon fixing the degree of the SOS multipliers, this is amount to solve an SDP.

### 3.3 Putinar's Positivstellensatz

The conditions in (8) and (9) are clearly sufficient conditions for the local nonnegativity of p(x) in (7). Do such representations always exist? The answer is in general no. However, under some mild conditions on p and S, we can guarantee the existence of such a representation. This is so-called Positivstellensatz results. We here only describe one of them.

**Theorem 14.5** (Putinar's Positivstellensatz). Let  $S = \{x \in \mathbb{R}^n \mid g_1(x) \ge 0, \dots, g_m(x) \ge 0\}$ . Assume there exists  $i \in \{1, \dots, m\}$  such that  $\{x \in \mathbb{R}^n \mid g_i(x) \ge 0\}$  is compact. Suppose p(x) is a polynomial such that  $p(x) > 0, \forall x \in S$ . Then, there exist SOS polynomials  $s_0(x), s_1(x), \dots, x_m(x)$  such that

$$p(x) = s_0(x) + \sum_{i=1}^m s_i(x)g_i(x).$$

There are some other Positivstellensatz that require different assumptions on S; see [3, Chpater 3].

### Notes

The preparation of this lecture was based on [2, Lectures 13 & 14] and [1, Chapter 3]. Further reading for this lecture can refer to [1, Chapter 3] and [3, Chapter 3].

## References

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