ECE285: Semidefinite and sum-of-squares optimization **Winter 2025**

Lecture 14: Nonnegative polynomials, SOS, and SDPs (II)

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Learning goals:

1. Nonnegative multivariate polynomials
2. Sum-of-squares hierarchies
3. Nonnegativity on Sets: Positivstellensatz

1 Nonnegative multivariate polynomials

In this lecture, we start to look at polynomials in more than one variable. We denote by $\mathbb{R}[x]$ the space of polynomials in n variables $x_1, \ldots, x_n$. A monomial is expressed as $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $x = (x_1, \ldots, x_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$, where $\alpha_i, i = 1, \ldots, n$ are integers. The degree of a monomial x^α is $|\alpha| := \alpha_1 + \alpha_2 + \ldots + \alpha_n$. The degree of a polynomial is the largest degree of its monomials. For example,

$$p(x) = x_1^3 + 2x_1x_2^2 + x_1x_2 + 2 \tag{1}$$

has degree 3. We denote by $\mathbb{R}[x]_{n,d}$ the space of polynomials in n variables of degree no bigger than d. The canonical monomial basis for $\mathbb{R}[x]_{n,d}$ is

$$[x]_d = \left[1, x_1, x_2, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_n^2, \ldots, x_1^d, \ldots, x_n^d\right]^{\mathsf{T}},$$

which has size $\binom{n+d}{d}$. Each element $p(x) = \sum_{|\alpha| \leq d} p_\alpha x^\alpha \in \mathbb{R}[x]_{n,d}$ can be identified by its coefficients $p_\alpha, |\alpha| \leq d$. For example, when $n = 2, d = 3$, we have

$$[x]_d = \left[1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\right]^{\mathsf{T}},$$

and the polynomial in (1) reads as

$$p(x) = \left[2, 0, 0, 0, 1, 0, 1, 0, 2, 0\right] [x]_d.$$

We are interested in polynomials $p(x)$ that are nonnegative globally on $\mathbb{R}^n$, i.e., $p(x) \geq 0, \forall x \in \mathbb{R}^n$. An obvious sufficient condition for nonnegative polynomials is a sum-of-squares (SOS) representation.

Definition 14.1. *A polynomial $p(x) \in \mathbb{R}[x]$ is a sum of squares if there exists polynomials $q_1(x), \ldots, q_t(x) \in \mathbb{R}[x]$ such that*

$$p(x) = q_1(x)^2 + \ldots + q_t(x)^2.$$

It is not difficult to see that if $p(x) \in \mathbb{R}[x]$ is nonnegative globally, it must has even degree. If $p(x)$ is of even degree $2d$ and $p(x) = \sum_{k=1}^t q_k(x)^2$, then we always have $\deg(q_k) \leq d, k = 1, \ldots, t$.

In Lecture 11, we have seen that for univariate polynomials ($n = 1$), nonnegativity is equivalent to an SOS condition. It turns out that in general this is not true. In fact, for polynomials of degree no less than four, deciding whether a polynomial is nonnegative globally is NP-hard (as a function of the number of variables).

Let $P_{n,2d}$ be the cone of nonnegative polynomials in n variables of degree at most $2d$. Let $\Sigma_{n,2d}$ be the cone of SOS polynomials in n variables of degree at most $2d$. More than a century ago, David Hilbert showed the equivalence between nonnegative polynomials and SOS polynomials (i.e., $\Sigma_{n,2d} = P_{n,2d}$) holds only in the following three cases.

Theorem 14.1 (Hilbert)**.** $\Sigma_{n,2d} = P_{n,2d}$ *holds only in the following three cases:*

- $n = 1$ *(univariate polynomials);*
- $2d = 2$ *(quadratic polynomials);*
- $n = 2, 2d = 4$ *(bivariate quartics).*

We have already seen that $\Sigma_{n,2d} = P_{n,2d}$ in the case of $n = 1$. The case $2d = 2$ can be proved easily using the eigenvalue decomposition of positive semidefinite matrices. The last case $n = 2, 2d = 4$ is more difficult.

Checking $p(x) \in P_{n,2d}$ is computationally hard. On the other hand, checking $p(x) \in \Sigma_{n,2d}$ can be performed using semidefinite programs.The following result is a multivariate generalization of the result in the previous lecture. For convenience, we write $s(n, d) = \binom{n+d}{d}$.

Theorem 14.2. *Consider a polynomial of degree* $2d$ *in the form of*

$$p(x) = \sum_{|\alpha| \leq 2d} p_\alpha x^\alpha.$$

Then $p(x) \in \Sigma_{n,2d}$ *if and only if there exists a positive semidefinite matrix* $Q \in \mathbb{S}_+^{s(n,d)}$ *such that*

$$p_\alpha = \sum_{\beta+\gamma=\alpha} Q_{\beta\gamma}, \quad \forall |\alpha| \leq 2d. \tag{2}$$

Proof. We first prove "Only if" part. Suppose $p(x) \in \Sigma_{n,2d}$. There exist $q_1(x), \ldots, q_t(x) \in \mathbb{R}[x]_{n,d}$ such that

$$p(x) = \begin{bmatrix} q_1(x) \\ \vdots \\ q_t(x) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} q_1(x) \\ \vdots \\ q_t(x) \end{bmatrix}. \tag{3}$$

Upon choosing the monomial basis $[x]_d$ and collecting all the coefficients of $q_1(x), \ldots, q_t(x)$, we have

$$\begin{bmatrix} q_1(x) \\ \vdots \\ q_t(x) \end{bmatrix} = V[x]_d.$$

Then, (3) becomes

$$p(x) = (V[x]_d)^{\mathsf{T}} V[x]_d = [x]_d \left(V^{\mathsf{T}} V\right) [x]_d.$$

Therefore, we obtain a positive semidefinite matrix $Q = V^{\mathsf{T}} V \in \mathbb{S}_+^{s(n,d)}$. The linear equations are obtained by matching coefficients

$$p(x) = [x]_d Q [x]_d = \sum_{|\alpha| \leq 2d} \left(\sum_{\beta+\gamma=\alpha} Q_{\beta\gamma} \right) x^\alpha.$$

"If" part is clear by reversing the arguments above based on a Cholesky factorization of $Q = V^{\mathsf{T}} V$. $\square$

Remark 14.1. *Similar to the univariate case, deciding the membership* $p(x) \in \Sigma_{n,2d}$ *or performing linear optimization over* $\Sigma_{n,2d}$ *is equivalent to solving an SDP.* $\square$

Example 14.1 ([1, Example 3.38]). *Consider a multivariate polynomial*

$$p(x,y) = 2x^4 + 5y^4 - x^2y^2 + 2x^3y + 2x + 2.$$

We want to find an SOS representation for this polynomial. In this case, $n = 2$ and $2d = 4$, the monomial basis is

$$[x]_d = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 \end{bmatrix}^\top,$$

and the matrix Q, indexed by this monomial basis, is

$$Q = \begin{bmatrix} q_{00,00} & q_{00,10} & q_{00,01} & q_{00,20} & q_{00,11} & q_{00,02} \\ & q_{10,10} & q_{10,01} & q_{10,20} & q_{10,11} & q_{10,02} \\ & & q_{01,01} & q_{01,20} & q_{01,11} & q_{01,02} \\ & & & q_{20,20} & q_{20,11} & q_{20,02} \\ & & & & q_{11,11} & q_{11,02} \\ & & & & & q_{02,02} \end{bmatrix} \tag{4}$$

(The rest of entries are symmetrical.) Checking whether $p(x,y)$ is SOS is equivalent to deciding whether there is a matrix Q in (4) that is positive semidefinite and satisfies the linear constraints (2). In this case, there is a total of $s(n, 2d) = \binom{6}{4} = 15$ linear constraints, one for each monomial x^α of degree at most $2d$. For instance, the equations corresponding to monomials x^4, x^2y^2, and y^2 are

$$\begin{aligned} x^4 : &\qquad 2 = q_{20,20} \\ x^2y^2 : &\qquad -1 = 2q_{20,02} + q_{11,11} \\ y^2 : &\qquad 0 = 2q_{00,02} + q_{01,01}. \end{aligned}$$

This amounts to solving a feasibility of an SDP. One feasible solution is given as

$$Q = \frac{1}{3}\begin{bmatrix} 6 & 3 & 0 & -2 & 0 & -2 \\ & 4 & 0 & 0 & 0 & 0 \\ & & 4 & 0 & 0 & 0 \\ & & & 6 & 3 & -4 \\ & & & & 5 & 0 \\ & & & & & 15 \end{bmatrix}.$$

Any factorization of matrix Q will lead to an SOS decomposition. For instance, one SOS decomposition is

$$p(x,y) = \frac{4}{3}y^2 + \frac{1349}{705}y^4 + \frac{1}{12}(4x+3)^2 + \frac{1}{15}(3x^2+5xy)^2 + \frac{1}{315}(-21x^2+20y^2+10)^2 + \frac{1}{59220}(328y^2-235)^2.$$

□

Theorem 14.1 has identified all the cases where nonnegativity is equivalent to an SOS condition. For all other cases, there always exist nonnegative polynomials that are not sum-of-squares. Consider the Motzkin polynomial ($n = 2, 2d = 6$)

$$M(x,y) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2.$$

One can show that $M(x,y)$ is nonnegative globally via the arithmetic-geometric mean inequality:

$$\frac{1}{3}(x^4y^2 + x^2y^4 + 1) \geq (x^6y^6)^{1/3} = x^2y^2, \qquad \forall x, y \in \mathbb{R}.$$

On the other hand, one can show that $M(x,y)$ is not a sum of squares. In fact, one can prove that $M(x,y) - \gamma$ is not a sum of squares for any $\gamma \in \mathbb{R}$.

Proposition 14.1. *$M(x,y) - \gamma$ is not a sum of squares for any $\gamma \in \mathbb{R}$.*

2 Sum-of-squares hierarchies

Consider a global polynomial optimization problem

$$p^\star = \min_{x \in \mathbb{R}^n} \quad p(x) \tag{5}$$

where $p(x) \in \mathbb{R}[x]_{n,2d}$. As we have seen last time, this is equivalent to

$$\begin{aligned} \gamma^\star = \max_{\gamma} \quad & \gamma \\ \text{subject to} \quad & p(x) - \gamma \geq 0, \qquad \forall x \in \mathbb{R}^n. \end{aligned}$$

We can establish a lower bound by solving an SDP as

$$\begin{aligned} \gamma_0 = \max_{\gamma} \quad & \gamma \\ \text{subject to} \quad & p(x) - \gamma \in \Sigma_{n,2d}. \end{aligned}$$

We have $\gamma_0 \leq \gamma^\star = p^\star$. In the case of univariate polynomials ($n = 1$), we have $\gamma_0 = \gamma^\star$ since nonnegative univariate polynomials are SOS. However, we have a strict inequality in general. Recall the Motzkin polynomial $M(x, y) - \gamma$ is not SOS for any γ. In this case, $\gamma_0 = -\infty$. This lower bound from the semidefinite relaxation is not useful. On the other hand, we can verify that

$$\begin{aligned} (1 + x^2 + y^2)M(x, y) =& y^2(1 - x^2)^2 + x^2(1 - y^2)^2 + (x^2y^2 - 1)^2 \\ &+ x^2y^2 \left(\frac{3}{4}(x^2 + y^2 - 2)^2 + \frac{1}{4}(x^2 - y^2)^2\right). \end{aligned}$$

This also clearly shows that $M(x, y) \geq 0, \forall (x, y) \in \mathbb{R}^2$.

The fact above actually suggests a hierarchy of sum-of-squares relaxations for (5)

$$\begin{aligned} \gamma_r = \max_{\gamma} \quad & \gamma \\ \text{subject to} \quad & (1 + x_1^2 + \ldots + x_n^2)^r (p(x) - \gamma) \in \Sigma_{n,2d+2r}. \end{aligned} \tag{6}$$

It is not hard to show that $\gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \ldots \leq p^\star$. Note that one can define another hierarchy of semidefinite relaxations where the multiplier $(1 + x_1^2 + \ldots + x_n^2)^r$ is replaced by another nonnegative polynomial. This will lead to a different hierarchy.

One natural question is to ask whether the sequence γ_r defined in (6) converges to $p^\star$. Under some assumptions on $p(x)$, one can actually establish the convergence result. For example, we have the following result, established by Reznick [4] (a homogeneous polynomial of degree $2d$ is a polynomial with only monomials of degree exactly $2d$, e.g., $p(x_1, x_2) = x_1^4 + x_2^4 + x_1^2x_2^2$).

Theorem 14.3 (Reznick's theorem). *Let $p(x) \in \mathbb{R}[x]_{n,2d}$ be a homogeneous polynomial. If $p(x) > 0, \forall x \in \mathbb{R}^n \setminus \{0\}$, then there exists $r \in \mathbb{N}$ such that*

$$(x_1^2 + \ldots x_n^2)^r p(x)$$

is a sum of squares.

Note that $(x_1^2 + \ldots x_n^2)^r p(x)$ being SOS means that $p(x)$ can be written as a sum of squares of rational functions. Hilbert's 17th problem asks whether any nonnegative polynomial can be written a sum of squares of rational functions. This question was answered positively by Artin in 1927.

Theorem 14.4 (Hilbert–Artin's theorem). *Let $p(x) \in \mathbb{R}[x]_{n,2d}$. If $p(x) \geq 0, \forall x \in \mathbb{R}^n$, then there exist nonzero SOS polynomials $h(x)$ and $q(x)$ such that*

$$h(x)p(x) = q(x).$$

3 Nonnegativity on Sets

An SOS representation is an obvious certificate of the global nonnegativity of a polynomial $p(x)$ over the entire space $\mathbb{R}^n$. In this section, we look into the nonnegativity of a polynomial $p(x)$ on a given subset $S \in \mathbb{R}^n$, i.e., we aim to provide certificates for

$$p(x) \geq 0, \quad \forall x \in S. \tag{7}$$

The set S could be defined in different forms, and the certificates we discuss below will depend on how S is presented.

3.1 Equations

Consider the set S is defined by a set of polynomial equations

$$S = \{x \in \mathbb{R}^n \mid f_1(x) = 0, \ldots, f_m(x) = 0\}.$$

Recalling the form of Lagrange multipliers, we can write the following condition

$$p(x) + \sum_{i=1}^{m} \lambda_i(x) g_i(x) \quad \text{is SOS}, \tag{8}$$

where $\lambda_i(x)$ are arbitrary polynomials. If we can find $\lambda_i(x)$ such that (8) holds, then it is obviously a certificate to establish (7). Indeed, if (8), by evaluating this expression at any point $x_0 \in S$, we conclude that $p(x_0) \geq 0$. Note that (8) is affine in the unknown polynomials $\lambda_i(x)$. Once we fix the degree of $\lambda_i(x)$, (8) is amount to solving an SDP.

3.2 Inequalities

Consider the set S is defined by a set of polynomial inequalities

$$S = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}.$$

We have very similar arguments. Inspired by Lagrange multipliers, we consider

$$p(x) = s_0(x) + \sum_{i=1}^{m} s_i(x) g_i(x), \tag{9}$$

where $s_0(x)$ and $s_i(x)$ are SOS polynomials. This condition (9) serves a obvious certificate for (7). Indeed, evaluating the expression (9) at any point $x_0 \in S$ directly proves $p(x_0) \geq 0$. Again, (9) is affine in the unknown polynomials $s_0(x), s_i(x)$. Once we fix the degree of $s_0(x), s_i(x)$, (9) is amount to solving an SDP.

Remark 14.2. *We can also consider more powerful expressions by allowing finite products in the following form*

$$p(x) = s_0(x) + \sum_{i=1}^{m} s_i(x) g_i(x) + \sum_{i,j}^{m} s_{ij}(x) g_i(x) g_j(x) + \ldots$$

where $s_0(x), s_i(x), s_{ij}(x)$ are SOS polynomials. This representation is also an obvious certificate for (7)*. Upon fixing the degree of the SOS multipliers, this is amount to solve an SDP.* □

3.3 Putinar's Positivstellensatz

The conditions in (8) and (9) are clearly sufficient conditions for the local nonnegativity of $p(x)$ in (7). Do such representations always exist? The answer is in general no. However, under some mild conditions on p and S, we can guarantee the existence of such a representation. This is so-called Positivstellensatz results. We here only describe one of them.

Theorem 14.5 (Putinar's Positivstellensatz)**.** *Let $S = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$. Assume there exists $i \in \{1, \ldots, m\}$ such that $\{x \in \mathbb{R}^n \mid g_i(x) \geq 0\}$ is compact. Suppose $p(x)$ is a polynomial such that $p(x) > 0, \forall x \in S$. Then, there exist SOS polynomials $s_0(x), s_1(x), \ldots, x_m(x)$ such that*

$$p(x) = s_0(x) + \sum_{i=1}^{m} s_i(x) g_i(x).$$

There are some other Positivstellensatz that require different assumptions on S; see [3, Chpater 3].

Notes

The preparation of this lecture was based on [2, Lectures 13 & 14] and [1, Chapter 3]. Further reading for this lecture can refer to [1, Chapter 3] and [3, Chapter 3].

References

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