State space systems

① **Model** : state space model $\underline{1}, \underline{2}, \underline{3}, \underline{4}$

$\dot{x} = Ax + Bu$

$y = Cx$

$\begin{cases} \text{nonlinear equation} \xrightarrow[\text{Tylar}]{\text{linearization}} \text{ODEs} \\ \text{transfer function (Experimental identification)} \end{cases}$

1) nonlinear equation $C(sI-A)^{-1}B$ $\begin{cases} ① \text{ not easy } (sI-A)^{-1} \\ ② \text{ many element are irrelavent} \end{cases}$

$\dot{x} = f(x,u)$ (x_*, u_*) : equilibrium point $f(x_*, u_*) = 0$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$f(x,u) = f(x_*, u_*) + [\nabla f(x_*, u_*)]^T \begin{bmatrix} x - x_* \\ u - u_* \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\Bigg|_{x=x_*} \qquad B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}\Bigg|_{u=u_*}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & a_{n-1} \end{bmatrix}$$

2) transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \Leftrightarrow \frac{d}{dt}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = Ax + Bu \quad y = Cx$$

$B = [0, 0, \cdots, 1]^T$ $C = [1, 0, \cdots, 0]$

② **Analysis** : Free and force response $\underline{5}$

$u = 0$ $\quad x = e^{At}x_0 = \sum_{i=1}^{n} e^{\lambda_i t} w_i v_i^T x_i(0) = We^{\Lambda t}Vx(0)$ $\quad W$: eigenvectors

$V = W^{-1}$

$x = Pz$ $\quad \dot{z} = P^{-1}APz + P^{-1}Bu(t)$ (see Example 2.1)

$z_i = e^{\lambda_i t} z_i(0)$

$x = \sum_{i=1}^{n} w_i z_i = \sum_{i=1}^{n} e^{\lambda_i t} w_i z_i(0)$

$z_i(0) = P^{-1}x(0) = Vx(0)$

$x = \sum_{i=1}^{n} e^{\lambda_i t} w_i v_i^T x(0)$

③ **Properties** : Controllable, observable. $\underline{6}, \underline{7}, \underline{8}$

$\begin{cases} \text{physical meaning} \\ \text{Test} \quad [B \; AB \cdots, A^{n-1}B] \end{cases}$ $\quad \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$

④ **State Feedback** : Pole placement $\underline{9}$

$\dot{x} = Ax + Bu$ $\quad u = r - Gx$

Please see Example 4.1

$\dot{x} = (A - BG)x + Br$

$$|sI - (A - BG)| = |sI - A + Bfk^T| \qquad |I + qr^T| = 1 + q^T r$$

$$= |sI - A|(1 + k^T(sI - A)^{-1}Bf) = 0$$

$k^T q_i = -1$

$$\begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} k = \begin{bmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}, \quad k = Q^{-1}\begin{bmatrix} -1 \\ \vdots \\ -1 \end{bmatrix}$$

① Response of linear state-space systems

$\dot{x} = Ax + Bu$

① Free response : $u = 0$, $x = e^{At} x(0)$

Note: Just compute λ_i, W, V

then $x = \sum_{i=1}^{n} e^{\lambda_i t} w_i v_i^T x(0)$

$x = Wz \Rightarrow \dot{z} = W^{-1}AWz + W^{-1}Bu$

$W^{-1}AW = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, $z_i = e^{\lambda_i t} z_i(0)$

$x = W \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \sum_{i=1}^{n} w_i z_i = \sum_{i=1}^{n} e^{\lambda_i t} w_i z_i(0)$

$z(0) = W^{-1} x(0) = V x(0) \Rightarrow z_i(0) = v_i^T x(0)$

$x = \left(\sum_{i=1}^{n} e^{\lambda_i t} w_i v_i^T \right) x(0) = W e^{\Lambda t} V x(0)$

② Force response $u \neq 0$

$$x = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$$

② Pole Placement

$\dot{x} = Ax + Bu$ Feedback $u = r - Gx$

$\dot{x} = (A - BG)x + Br$ $G \in R^{m \times n}$: more flexibility $\Rightarrow G = fk^T$

$$|sI - (A - BG)| = |sI - A + Bfk^T| = |sI - A|\,|sI + [sI - A]^{-1} Bfk^T|$$

$$= |sI - A|(1 + k^T [sI - A]^{-1} Bf) = 0$$

desired poles : $s = p_i$,

$$1 + k^T \underbrace{[p_i I - A]^{-1} Bf}_{q_i} = 0 \quad , \ i = 1, \cdots, n$$

$Qk = \begin{bmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}$ $Q = \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix}$

Note: this is not easy to compute by hand. If you have this kind of problems in exams, you can directly compute $|sI - (A - BG)| = 0$

Normally $A \in R^{2 \times 2}$, $B \in R^{2 \times 1}$, $G \in R^{1 \times 2}$

State Space Systems

$$\frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \\ q \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ q \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} V$$

$$V_c = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ q \end{bmatrix}$$

① $G(s) = C(sI - A)^{-1}B$

②

$$\begin{cases} \dot{i}_1 = -i_1 + i_2 + V \\ \dot{i}_2 = i_1 - i_2 - q \\ \dot{q} = i_2 \end{cases} \Rightarrow \begin{aligned} sI_1 &= -I_1 + I_2 + V & ① \\ sI_2 &= I_1 - I_2 - Q & ② \\ sQ &= I_2 & ③ \end{aligned}$$

$$V_c = q \qquad\qquad V_c = Q \quad ④$$

$$I_2 = sQ = sV_c$$

$$I_1 = (s+1)I_2 + Q = s(s+1)V_c + V_c = (s^2+s+1)V_c$$

$$① \Rightarrow \quad (s+1)I_1 - I_2 = V$$

$$(s+1)(s^2+s+1)V_c - sV_c = V$$

$$G(s) = \frac{V_c}{V} = \frac{1}{(s+1)(s^2+s+1) - s} = \frac{1}{s^3 + 2s^2 + s + 1}$$