A brief summary of B15: Control systems

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May 9, 2018

Note: this is a brief summary of B15: control systems. The purpose is to help you review the main points. You should rely on the lecture notes for completeness.

There are in general four parts in this course: 1) state-space systems; 2) feedback control, which mainly talks about linear quadratic regulator (LQR); 3) Kalman fitering; and 4) limits of controller performance.

1 State-space systems

In this part, you should

(a) Be able to linearise nonlinear models to get a state space description (Taylor expansion);

(b) Derive the free and force response of linear systems;

(c) Determine structural properties: controllability and observability; stabilizability and detectability;

(d) Use state feedback to place system poles;

(e) Understand and design observers.

State-space models: a state-space model uses only a set of first-order differential equations to describe the behavior of a dynamical system.

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned} \tag{1}$$

where x, y, u denote the sate, output and control input, respectively.

ODEs: a linear ODE model describes the system behavior using higher-order derivatives. (See Section 1.3)

Nonlinear models: which use a set of nonlinear ODEs to model system behavior

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n, u), \\ \dot{x}_2 &= f_2(x_1, x_2, \ldots, x_n, u), \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \ldots, x_n, u), \end{aligned} \tag{2}$$

At the equilibrium point $x^0 = [x_1^0, x_2^0, \ldots, x_n^0]^T, u^0$, we have

$$f_i(x_1^0, x_2^0, \ldots, x_n^0, u^0) = 0, i = 1, \ldots, n.$$

Upon defining a new set of states (deviation) $\delta x_i = x_i - x_i^0, i = 1, \ldots, n; \delta u = u - u^0$, and using Taylor expansion, you should be able to derive a linearised state space model for (2)

$$\delta\dot{x} = A\delta x + B\delta u$$

where

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}, B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \vdots \\ \frac{\partial f_n}{\partial u} \end{bmatrix}.$$

(See Example 1.3 and Example 1.4 for relevant examples.)

Transfer functions: frequency domain. You should be able to do switch from transfer functions to state-space models and back. Usually, we assume zero initial condition, then a transfer function for (1) is

$$G(s) = C(sI - A)^{-1}B + D.$$

One difficulty is to calculate the inverse of $sI - A$.

Note that given a transfer function, its state space realization is **not unique**. They are related by a certain change of coordinates $x_1 = Tx_2$.

Exam hint: As we discussed in our tutorial, it might be easier to directly perform Laplace transform for each individual differential equation. In this way, we can rely on eliminations/substitutions and avoid doing an explicit inverse of $sI - A$.

Free response of a linear system: In (1), assume $u = 0$ and initial condition $x(0)$. The free response is

$$x(t) = e^{At}x(0) = We^{\Lambda t}Vx(0) = \sum_{i=1}^{n} e^{\lambda_i t} w_i v_i^T x_i(0),$$

where we have assumed the eigen-decomposition of A is

$$A = W \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} V.$$

(See Page 22-23 for some details about calculating an eigen-decomposition; also Example 2.1)
Structural properties:

- *Controllability:* System (1) is controllable if and only if the rank of $Q = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$ is n
- *Observability:* System (1) is observable if and only if the rank of $Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ is n
- *Stabilizability:* System (1) is stabilizable if and only if the uncontrollable modes are stable.
- *Detectability:* System (1) is detectable if and only if the unobservable modes are stable.

You should also understand their physical meaning. (See Page 29)

Exam hint: In general, it is much easier to use the rank test to check controllability/observability in your exam!

Pole placement: design a state feedback controller $u = -Kx$ such that the eigenvalues of the closed-loop system $\dot{x} = (A - BK)x$ locate at prescribed locations.
Please think about how to choose the closed-loop poles (see Page 45 for details).

Exam hint: since we usually deal with second-order systems with one input in the exam, one quick method is to parameterize $k = [k_1, k_2]$, and then write down the characteristic equation $$\lvert sI - (A - Bk) \rvert = 0,$$ where we have two unknown parameters k_1, k_2 and we have two conditions λ_1, λ_2. You can solve it.

Observers: In some cases, we cannot measure the full state, *i.e.* $C \neq I$ in (1). Then, we have to design a proper observer before applying the state feedback $u = -K\hat{x}$, where $\hat{x}$ is the state estimation.

$$\dot{\hat{x}} = A\hat{x} + Bu + K_o(y - C\hat{x}),$$

The closed-loop dynamics contain both system dynamics $A - BK$ and observer dynamics $A - K_oC$. In principle, the observer closed-loop poles (*i.e.*, $A - K_oC$) should be faster than those of $A - BK$.
(see Example 5.1 on Page 54)

2 Feedback control: LQR

In this part, you should

(a) Be able to obtain an LQR feedback gain via solving a Riccati equation.
(b) Be able to solve a simple Riccati equation by hand.
(c) Understand the property of an LQR controller, especially infinite gain margin and at least 60^0 phase margin.
(d) Know the conversion from a tracking problem to a regulator problem.

LQR problem: we are trying to solve the following constrained optimisation problem

$$\begin{aligned} \min_u \quad & J(x,u) = \frac{1}{2}\int_0^\infty [x(t)^T Q x(t) + u(t)^T R u(t)]dt \\ \text{subject to} \quad & \dot{x} = Ax + Bu \end{aligned} \tag{3}$$

The theory is very beautiful: the optimal solution to (3) is a linear state feedback

$$u = -Kx,$$

where $K = R^{-1}B^T P$ and P is the **positive definite** solution to the **Riccati equation**

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \tag{4}$$

See Page 81 for derivation if you are interested.

Exam hint: We usually deal with second-order system in the exam. So, we can simply parameterize

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}.$$

- Note that P should be symmetric, so we have $p_{12} = p_{21}$. There are *three* scalar parameters.
- If you expanding (4), you will get *three* scalar equations. Then you can solve it.
- Note that the solution to the Riccati equation is not unique. We should look for a positive definite P, *i.e.*, the eigenvalues of P are positive. For 2×2 matrices, it is equivalent to

  $$p_{11} > 0, p_{22} > 0, p_{11}p_{22} - p_{12}^2 > 0.$$

  You should use this property to ignore some solutions.
- The optimal cost to (3) depends on its initial condition

  $$J_{opt} = \frac{1}{2}x^T(0)Px(0)$$

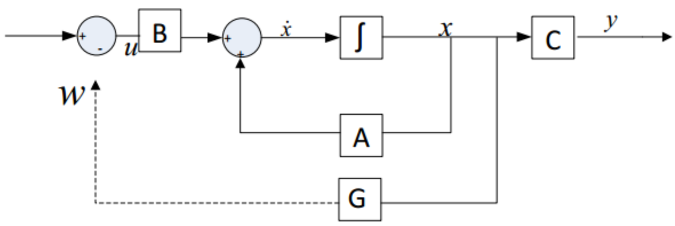

Figure 1: A state-space system controlled by state feedback

Loop transfer function: As shown in Fig. 1, the loop transfer function for an LQR controller is

$$L_{lqr}(s) = \frac{W(s)}{U(s)} = G(sI - A)^{-1}B \tag{5}$$

For the transfer function $L_{lqr}(s)$, we have a phase margin of 60^0 or more, and infinite gain margin.

Exam hint: If you are asked to verify the gain/phase margin of an LQR design. You should work out the expression of $L_{lqr}(s)$, and then find the corresponding gain/phase margin.

Tracking problem. Define the error state $\delta x = x_0 - x$, where x_0 is the desired trajectory. Then we have

$$\begin{aligned} \delta\dot{x} &= A\delta x + B\delta u, \\ \delta y &= C\delta x, \end{aligned}$$

which is now a regulation problem. See Page 87 for details.

3 Kalman filtering

In this part, you should

(a) Be able to obtain a Kalman gain via solving a Riccati equation.
(b) Understand the meaning of Kalman filter tuning parameters.
(c) Use the LQG design to derive optimal control laws (separation theorem).
(d) Understand the Loop transfer recovery to recover robustness of LQG control laws.

Kalman filtering: Find an optimal estimator for the linear system

$$\begin{aligned} \dot{x} &= Ax + Bu + Fv \\ y &= Cx + w \end{aligned} \tag{6}$$

The observer structure is

$$\dot{\hat{x}} = A\hat{x} + Bu + K_f(y - C\hat{x}). \tag{7}$$

The Kalman gain is given by

$$K_f = PC^TW^{-1},$$

and P is the solution to

$$\dot{P} = AP + PA^T - PC^TW^{-1}CP + FVF^T, \tag{8}$$

where W is the co-variance of sensor measurement w, V is the covariance of dynamical noise v.

Exam hint: Usually, we care about the steady Kalman gain, then $\dot{P} = 0$. We only need to solve a Riccati equation

$$AP + PA^T - PC^TW^{-1}CP + FVF^T = 0.$$

This can be viewed as a dual concept of LQR. (See the lecture notes on duality between LQR and Kalman filtering).
Also, the Kalman gain K_f usually depends on the 'signal-to-noise' ratio. (See Example 3.2)

LQG design: we are trying to solve the following constrained optimisation problem

$$\begin{aligned} \min_u \quad & J = \frac{1}{2}E\left[\int_0^\infty (x^TQx + u^TRu)\right] dt \\ \text{subject to} \quad & \dot{x} = Ax + Bu + Fv \\ & y = Cx + w \end{aligned} \tag{9}$$

The theory is very beautiful: we have the *separation principle*. The optimal control law to (9) is the LQR feedback applied to the Kalman estimation $\hat{x}$

$$u = -G\hat{x},$$

where $\hat{x}$ is the state estimation from (7).

Loop transfer recovery (LTR): One major issue is that the LQG design cannot guarantee stability margin. One simple method to recover the robustness is the LTR procedure:

1. Choose a q and set $V = V_0 + q^2I$. Use the Riccati equation to calculate a Kalman gain K_f;
2. The LQG loop transfer function is

   $$L_{lqg}(s) = G(sI - A + BG + K_fC)^{-1}K_f \times C(sI - A)^{-1}B.$$

   check the gain and phase margins of the close-loop system.
3. Once the gain and phase margins are satisfactory. Check the sensitivity to sensor noise.

Since in theory one can prove

$$\lim_{q\to\infty} L_{lqg}(s) = L_{lqr}(s),$$

we can use the LTR procedure to improve the robustness of LQG controller. Note that if q is too large, the system will be very sensitive to sensor noise.

Exam hint: You need to understand the principle of the LTR procedure, and you should be able to write down the steps roughly. It is unlikely to have actual calculations in the exam. (see my code for numerical simulations)

4 Limits of controller performance

In this part, you should

(a) Be able to describe a system using an ODE, transfer function, impulse response and state space model, and be able to switch between them.

(b) Have a deeper understanding of closed-loop stability; BIBO stability.

(c) Understand how to draw Nyquist plot, and Nyquist stability criterion.

(d) Appreciate the limitations on performance (particularly Bode integral theorem).

(e) Understand the basic robustness measures, particularly $\mathcal{H}_\infty$ norm.

Model descriptions: similar to Part I. Recall that the key point between different description is *The roots of the characteristic equation of the ODE* are the same as *the poles of the transfer function* are the same as *the eigenvalues of the matrix A in the state space model.*

Stability and BIBO stability: A system is stable if *the root/pole/eigenvalue* lies in the *open left half plane.* A transfer function $G(s)$ is BIBO stable if the integration of its impulse response is finite

$$\int_0^\infty |g(\tau)| d\tau < \infty.$$

Exam hint: It is usually easier to use the partial fraction expansion to derive a impulse response. (inverse Laplace transform).

$$G(s) = \frac{R_1}{s - p_1} + \frac{R_2}{s - p_2} + \ldots + \frac{R_n}{s - p_n},$$

and its impulse response is given by

$$g(t) = R_1 e^{p_1 t} + R_2 e^{p_2 t} + \ldots + R_n e^{p_n t}.$$

Nyquist Stability Criterion: A closed-loop system will be stable if and only if the mapping of the D-contour under the open-loop transfer function $L(s) = G(s)C(s)$ encircles the point $-1 + j0$ in an anti-clockwise direction P times, where P is the number of poles of $L(s)$ in the right half plane.

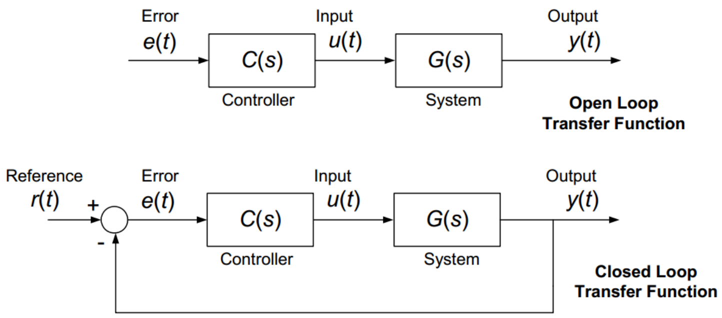

Figure 2: Block diagrams of open-loop and closed-loop systems

Exam hint: Nyquist plot is an *indirect* method to check closed-loop stability. As shown in Fig. 2, we are looking into the property of the open-loop system $L(s) = G(s)C(s)$ to determine the stability of closed-loop system

$$T(s) = \frac{G(s)C(s)}{1 + G(s)C(s)}.$$

- Please remember that you should draw the Nyquist plot for the open-loop transfer function.
- Check the lecture note on the steps of drawing a Nyquist plot. The D-contour may have three parts (positive imaginary axis, negative imaginary axis, and big semi-circle) or four parts (positive imaginary axis, negative imaginary axis, big semi-circle, and small semi-circle), depending on whether the open-loop system has unstable poles at $s = 0$.
- Nyquist plot can be also used to determine phase margin and gain margin.
- For checking stability only, you can also compute the closed-loop transfer function and look at the closed-loop poles directly.

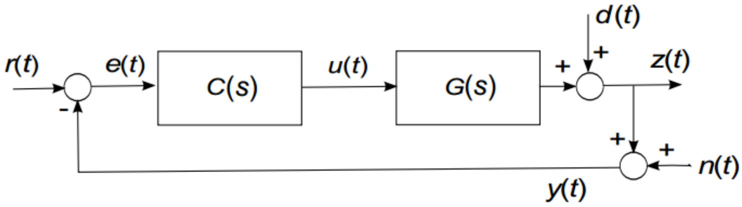

Figure 3: Signals in a closed-loop system

Closed-loop performance: As shown in Fig. 3, sensitivity transfer function is

$$S(s) = \frac{1}{1 + G(s)C(s)},$$

and complementary sensitivity transfer function is

$$T(s) = \frac{G(s)C(s)}{1 + G(s)C(s)},$$

and the response

$$Z(s) = T(s)R(s) + S(s)D(s) - T(s)N(s).$$

Since we want to reject the influence of disturbances, it is desirable to make $|S(j(w))|$ small. However, there is a certain limit depending the open loop unstable poles (Bode integral theorem)

$$\int_0^\infty \ln|S(jw)|dw = \pi \sum_{k=1}^{M} \mathrm{Re}(p_k^u).$$

System norms and uncertainty: the $\mathcal{H}_\infty$ norm of a transfer function $G(s)$ is

$$\|G(s)\|_\infty = \sup_\omega |G(j\omega)| = \sup_{\|u(t)\|_2 \neq 0} \frac{\|Gu\|_2}{\|u\|_2},$$

where the latter expression is an induced norm. For a scalar transfer function, we usually use the former expression to compute its $\mathcal{H}_\infty$ norm.

Check the lecture notes.