Optimal control and Convex optimization **Spring 2020**

Lecture 1: Problem formulation

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications. They were developed when the author was a postdoc in Prof. Na Li's group at Harvard. Any typos should be sent to* `zhengy@eng.ucsd.edu`.

Learning goals:

1. Problem formulation of optimal control;
2. State-space model of the closed-loop system;
3. Well-posedness of feedback systems;
4. Internal stability.

1 Problem formulation

We consider continuous-time linear time-invariant (LTI) systems of the form

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \\ z &= C_1 x + D_{11} w + D_{12} u, \\ y &= C_2 x + D_{21} w + D_{22} u, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, w \in \mathbb{R}^d, y \in \mathbb{R}^p, z \in \mathbb{R}^q$ are the state vector, control action, external disturbance, measurement, and regulated output, respectively. System (1) can be written as

$$\mathbf{P} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix},$$

where $\mathbf{P}_{ij} = C_i(sI - A)^{-1}B_j + D_{ij}$. We refer to $\mathbf{P}$ as the open-loop plant model.

Consider a dynamic output feedback controller $\mathbf{u} = \mathbf{K}\mathbf{y}$, where $\mathbf{K}$ has a state-space realization

$$\begin{aligned} \dot{\xi} &= A_k \xi + B_k y, \\ u &= C_k \xi + D_k y, \end{aligned} \tag{2}$$

where $\xi \in \mathbb{R}^{n_k}$ is the internal state of controller $\mathbf{K}$. We have $\mathbf{K} = C_k(sI - A_k)^{-1}B_k + D_k$. Figure 1 shows a schematic diagram of the interconnection of plant $\mathbf{P}$ and controller $\mathbf{K}$.

Problem formulation: Optimal control
Informally speaking, we aim to find a controller $\mathbf{K}$ such that the closed-loop system is internally stable and achieves/minimizes desired performance specification:

$$\begin{aligned} \min_{\mathbf{K}} \quad & f(\mathbf{P}, \mathbf{K}) \\ \text{subject to} \quad & \mathbf{K} \text{ internally stabilizes } \mathbf{P}. \end{aligned} \tag{3}$$

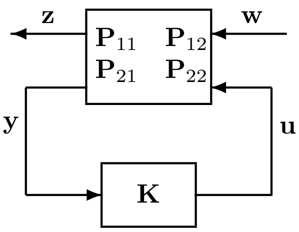

Figure 1: Interconnection of the plant $\mathbf{P}$ and controller $\mathbf{K}$

where $f(\mathbf{P},\mathbf{K})$ defines a certain performance index.

In (3), the performance index $f(\mathbf{P},\mathbf{K})$ quantifies the influence of the disturbance w on the performance output z, which is usually captured by a certain norm ($\mathcal{H}_2$ or $\mathcal{H}_\infty$) of the closed-loop response from w to z.

1.1 Closed-loop system in the frequency domain

By (1), we have

$$\mathbf{z} = \mathbf{P}_{11}\mathbf{w} + \mathbf{P}_{12}\mathbf{u},$$
$$\mathbf{y} = \mathbf{P}_{21}\mathbf{w} + \mathbf{P}_{22}\mathbf{u}.$$

Considering the controller $\mathbf{u} = \mathbf{K}\mathbf{y}$, some simple algebra leads to

$$\mathbf{z} = (\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21})\mathbf{w}. \tag{4}$$

Thus, the closed-loop response from $\mathbf{w}$ to $\mathbf{z}$ is

$$\mathbf{T}_{zw} = \mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}.$$

In (3), the cost function is typically chosen as

$$f(\mathbf{P},\mathbf{K}) = \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}\|,$$

where $\|\cdot\|$ can be chosen as the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm.

In (4), one question to ask is when the inverse $(I - \mathbf{P}_{22}\mathbf{K})^{-1}$ exists. This is related to the the notion of *well-posedness* of feedback systems, which will be defined later.

1.2 Closed-loop system in the state-space domain

We can also derive the closed-loop system in the state-space domain, which is a state-space realization of (4). Combining (1) with (2) leads to

$$\frac{d}{dt}\begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_k \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ B_k & 0 \end{bmatrix}\begin{bmatrix} y \\ u \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w, \tag{5a}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C_2 & 0 \\ 0 & C_k \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} 0 & D_{22} \\ D_k & 0 \end{bmatrix}\begin{bmatrix} y \\ u \end{bmatrix} + \begin{bmatrix} D_{21} \\ 0 \end{bmatrix} w, \tag{5b}$$

$$z = \begin{bmatrix} C_1 & 0 \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} 0 & D_{12} \end{bmatrix}\begin{bmatrix} y \\ u \end{bmatrix} + D_{11} w \tag{5c}$$

From (5b), we have

$$\begin{bmatrix} I & -D_{22} \\ -D_k & I \end{bmatrix}\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C_2 & 0 \\ 0 & C_k \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} D_{21} \\ 0 \end{bmatrix} w, \tag{6}$$

This equation has a unique solution if and only if the following matrix

$$\begin{bmatrix} I & -D_{22} \\ -D_k & I \end{bmatrix}$$

is invertible, which is equivalent to that $I - D_{22}D_k$ or $I - D_kD_{22}$ is invertible[1]. Note that the matrix dimension $D_{22} \in \mathbb{R}^{p\times m}$ and $D_k \in \mathbb{R}^{m\times p}$. This is also the condition of well-posedness (there are other equivalent definitions of well-posedness).

Definition 1. *A feedback system is said to be well-posed if the solutions $u(t)$ and $y(t)$ are unique, given any initial condition $x(0)$ and $\xi(0)$ and $w(t) = 0, \forall t > 0$.*

Lemma 1. *The feedback system in Figure 1 is well-posed if and only if $I - D_{22}D_k$ is invertible.*

For simplicity, we make the following assumption.

Assumption 1. *It is assumed that the plant is strictly proper,* i.e. *$D_{22} = 0$. By Lemma 1, this guarantees that the closed-loop system is always well-posed.*

Now, (6) becomes

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C_2 & 0 \\ D_kC_2 & C_k \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} D_{21} \\ D_kD_{21} \end{bmatrix} w$$

Substituting this into (7) leads to

$$\frac{d}{dt}\begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} A + B_2D_kC_2 & B_2C_k \\ B_kC_2 & A_k \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B_1 + B_2D_kD_{21} \\ B_kD_{21} \end{bmatrix} w, \tag{7a}$$

$$z = \begin{bmatrix} C_1 + D_{12}D_kC_2 & D_{12}C_k \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} + (D_{11} + D_{12}D_kD_{21})w. \tag{7b}$$

This is a state-space version of the closed-loop response from w to z. We can write

$$\mathbf{T}_{zw} = \left[\begin{array}{cc|c} A + B_2D_kC_2 & B_2C_k & B_1 + B_2D_kD_{21} \\ B_kC_2 & A_k & B_kD_{21} \\ \hline C_1 + D_{12}D_kC_2 & D_{12}C_k & D_{11} + D_{12}D_kD_{21} \end{array}\right].$$

[1] This can be easily seen from the fact $\begin{bmatrix} I & 0 \\ D_k & I \end{bmatrix}\begin{bmatrix} I & -D_{22} \\ -D_k & I \end{bmatrix} = \begin{bmatrix} I & -D_{22} \\ 0 & I - D_kD_{22} \end{bmatrix}$.

Consider the special case of static output feedback control $u = D_k y$. The closed-loop matrix is $A + B_2 D_k C_2$, and we have

$$\mathbf{T}_{zw} = \left[\begin{array}{c|c} A + B_2 D_k C_2 & B_1 + B_2 D_k D_{21} \\ \hline C_1 + D_{12} D_k C_2 & D_{11} + D_{12} D_k D_{21} \end{array}\right].$$

1.3 Internal stability

We now define the fundamental notion of internal stability.

Definition 2. *The system in Fig. 1 is* internally stable *if it is well-posed, and the states $(x(t), \xi(t))$ converge to zero as $t \to \infty$ for all initial states $x(0), \xi(0)$ when $w(t) = 0, \forall t$.*

Lemma 2. *The system in Fig 1 is internally stable if and only if*

$$\hat{A} := \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix}$$

is stable.

The set of all stabilizing controllers is defined as

$$\mathcal{C}_{\text{stab}} := \{\mathbf{K} \mid \mathbf{K} \text{ internally stabilizes } \mathbf{P}\}. \tag{8}$$

It is well-known that $\mathcal{C}_{\text{stab}}$ is non-convex and it is not difficult to find explicit examples where $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{C}_{\text{stab}}$ and $\frac{1}{2}(\mathbf{K}_1 + \mathbf{K}_2) \notin \mathcal{C}_{\text{stab}}$. Lemma 2 leads to an explicit state-space characterization of the set $\mathcal{C}_{\text{stab}}$ as follows:

$$\mathcal{C}_{\text{stab}} = \left\{ \mathbf{K} \mid \hat{A} := \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} \text{ is stable} \right\}, \tag{9}$$

where $\mathbf{K} = C_k(zI - A_k)^{-1} B_k + D_k$. Unfortunately, the stability condition on A_{cl} in (9) is still non-convex in terms of the parameters (A_k, B_k, C_k, D_k).

2 Optimal control

Now, the optimal controller synthesis problem (3) can be precisely written as

$$\begin{aligned} \min_{\mathbf{K}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}\| \\ \text{subject to} \quad & \mathbf{K} \in \mathcal{C}_{\text{stab}}. \end{aligned} \tag{10}$$

The state-space version is

$$\begin{aligned} \min_{A_k, B_k, C_k, D_k} \quad & \left\| \left[\begin{array}{cc|c} A + B_2 D_k C_2 & B_2 C_k & B_1 + B_2 D_k D_{21} \\ B_k C_2 & A_k & B_k D_{21} \\ \hline C_1 + D_{12} D_k C_2 & D_{12} C_k & D_{11} + D_{12} D_k D_{21} \end{array}\right] \right\| \\ \text{subject to} \quad & \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} \text{ is stable.} \end{aligned} \tag{11}$$

Both (10) and (11) are non-convex in its present form.

The rest of topics will include

1. Performance specification: $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of transfer matrices and their computations via convex optimization (LMIs).
2. Convex reformulation of (10) in the frequency domain (Youla parameterization [9], system-level synthesis [1, 8], and input-output parameterization [3, 10]).
3. Convex reformulation of (11) in the state-space domain (convex optimization via LMIs) [6, 7].
4. Analytical solutions via solving Algebraic Riccati Equation (ARE) [2, 11].
5. Distributed control when introducing a subspace constraint on the controller $\mathbf{K} \in \mathcal{S}$ (Quadratic Invariance [5], Sparsity Invariance [4], etc.).

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